New Methods of Approximation of Step Functions

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Abstract—New methods of approximation of step functions with an estimation of the error of the approximation are suggested. The suggested methods do not have any of the disadvantages of traditional approximations of step functions by means of Fourier series and can be used in problems of mathematical modeling of a wide range of processes and systems.

Keywords: step functions, mathematical modeling, approximation, convergence, estimation of error, examples of application.

1. INTRODUCTION

Step functions are widely applied in various areas of scientific research. Technical and mathematical disciplines, such as automatic control theory, electrical and radio engineering, information and signal transmission theory, equations of mathematical physics, theory of vibrations, and differential equations are traditional fields of application [1-3].

Systems with step parameters and functions are considered highly nonlinear structures to emphasize the complexity of obtaining solutions for such structures. Despite the simplicity of step functions in segments, the construction of solutions in problems with step functions on the whole domain of definition requires using special mathematical methods, such as the alignment method [4] with the coordination of the solution by segments and switching surfaces. Generally, application of the alignment method requires overcoming substantial mathematical difficulties, and intricate solutions represented by complex expressions are obtained rather often.

In many cases, researchers rely upon approximation methods using Fourier series

 $f = \sum_{k=1}^{\infty} c_k \varphi_k$, where $\{\varphi_1, \varphi_2, \dots, \varphi_n, \dots\}$ is an orthogonal system in functional Hilbert space

 $L_2[-\pi,\pi]$ of measurable functions with Lebesgue integrable squares, $f \in L_2[-\pi,\pi], c_k = (f \cdot \varphi_k) / \| \varphi_k \|^2$. The trigonometric system of 2π periodic functions $\{1, \sin nx, \cos nx; n \in N\}$ is often taken as an orthogonal system. In this case, the following is fulfilled in the vicinity of discontinuity points $O_{\delta}(x_0) \sup_{x \in O_{\delta}(x_0)/\{x_0\}} |f(x) - S_n(x)| \longrightarrow A \neq 0$

, where $S_n(x)$ is the partial sum of the Fourier series. It is how Gibbs' phenomenon shows itself [5]. Thus, in the case of a function

$$f_0(x) = \operatorname{sign}(\sin x) \tag{1}$$

the point $x = \pi/m$, where m = 2[(n+1)/2], and [A] is the integral part of the number A, is the maximum point of the partial sum $S_n(f_0)$ of the trigonometric Fourier series [6] with

$$S_n(f_0, \pi/m) \xrightarrow[n \to \infty]{} \frac{2}{\pi} \int_0^{\pi} \frac{\sin t}{t} dt \approx 1,17898,$$

i.e., the absolute error value $\left| f_0(\pi/m) - \lim_{n \to \infty} S_n(f_0, \pi/m) \right| > 0$. It should be noted that $x = \pi/m \to 0+0$.

The graph of the partial sum $S_{20}(f_0)$ of the trigonometric series on the interval $[-\pi, \pi]$, which illustrates the presence of the Gibbs phenomenon is presented in Fig. 1.



Fig. 1. Presence of the Gibbs phenomenon

What is unpleasant in this case is that the Gibbs effect is generic and is present for any function $f \in L_2[a, b]$, which has limited variation on the interval [a, b], with isolated discontinuity point $x_0 \in (a, b)$. The following condition is fulfilled for such functions [6]

$$\lim_{n \to \infty} S_n(f, x_0 + \pi/m) = f(x_0 + 0) + \frac{d}{2} \cdot \left(\frac{2}{\pi} \int_0^{\pi} \frac{\sin t}{t} dt - 1\right), \text{ where } d = f(x_0 + 0) - f(x_0 - 0).$$

We show that absolute $\Delta = \Delta(x)$ and relative $\delta = \delta(x)$ errors of approximation in the vicinity of discontinuity points may be as large as we please. In fact,

$$\begin{split} \lim_{n \to \infty} \Delta(x_0 + \pi/m) &= \lim_{n \to \infty} \left| S_n(f, x_0 + \pi/m) - f(x_0 + \pi/m) \right| = \left| \lim_{n \to \infty} S_n(x_0 + \pi/m) - \lim_{n \to \infty} f(x_0 + \pi/m) \right| = \\ &= \left| f(x_0 + 0) + \frac{d}{2} \cdot \left(\frac{2}{\pi} \int_0^{\pi} \frac{\sin t}{t} dt - 1 \right) - f(x_0 + 0) \right| = \left| \frac{d}{2} \cdot \left(\frac{2}{\pi} \int_0^{\pi} \frac{\sin t}{t} dt - 1 \right) \right| = \Delta(d). \end{split}$$
The function
$$\Delta(d) \quad \text{is an infinitely large value, as}$$

$$\forall M > 0 \exists d = d^*(M) > 0 \; \forall d : |d| > d^* \Rightarrow \Delta(d^*) = \left| \frac{d^*}{2} \cdot \left(\frac{2}{\pi} \int_0^{\pi} \frac{\sin t}{t} dt - 1 \right) \right| > M. \text{ Such expression}$$

as $\left[2M\pi / \left(2\int_{0}^{\pi} \frac{\sin t}{t} dt - \pi \right) \right] + 1$, where [A] is the integral part of the number A, may be taken as d^{*} .

The proof is identical for the relative error $\delta(x) = \Delta(x)/|f(x)|$. Moreover, even when $d \in \mathbf{R}$ $(d \neq 0)$ is fixed for any M > 0, the function $f(x) \in L_2[a,b]$ may be selected in such a way that $\delta(x_0 + 0, d) = \Delta(x_0 + 0, d)/|f(x_0 + 0)| > M$. The function with $|f(x_0 + 0)| < \Delta(x_0 + 0, d)/M$, $f(x_0 + 0) \neq 0$ may be taken as an example for this case.

It should be noted that it is not necessary for the Fourier series to converge at each point even on the set of continuous functions $C[-\pi, \pi]$, which is commonly known.

The presence of the Gibbs phenomenon leads to extremely negative consequences of the use of the partial sum of a trigonometric series as an approximating function in fields such as radio engineering and signal transmission.

2. DESCRIPTION OF THE METHOD

In order to eliminate the mentioned disadvantages, new methods of approximation of step functions based on the use of trigonometric expressions represented by recursive functions are suggested in the present paper.

For example, consider the step function (1) in more detail. This function is often used as an example of the application of Fourier series, and, therefore, it is convenient to take this function for comparative analysis of a traditional Fourier series expansion and the suggested method.

Expansion of (1) into Fourier series has all the above mentioned disadvantages. In order to eliminate them, it is proposed to approximate the initial step function by a sequence of recursive periodic functions

$$\left\{ f_{n}(x) \mid f_{n}(x) = \sin\left((\pi/2) \cdot f_{n-1}(x)\right), f_{1}(x) = \sin x; n-1 \in \mathbb{N} \right\} \subset C^{\infty}[-\pi,\pi]$$
(2)

Graphs of the initial function (a thickened line) and its five successive approximations for this case are presented in Fig. 2. It can be seen that, even when n values are relatively small in the iterative procedure (2), the graph of the approximating functions approximates the initial function (1) rather well. In addition, approximating functions obtained using the suggested method do not have any of the disadvantages of Fourier series expansion. There is absolutely no sign of the Gibbs phenomenon.



Fig.2 Graphs of the initial function and its successive approximation

Certain peculiarities of the proposed approximating iterative procedure are to be mentioned.

It should be noted that functions $f_n(x)$ and $f_0(x)$ are uneven and periodic ones with a period of 2π . Functions $f_n(x+\pi/2)$ and $f_0(x+\pi/2)$ are even and periodic. Therefore, it is sufficient to consider the sequence of approximating functions (2) on the interval $[0, \pi/2]$.

Let $\{f_n(x)\} \subset L_2[0, \pi/2]$ and $f_0(x) \in L_2[0, \pi/2]$. As $\sup_{n \in N} \sup_{x \in [0, \pi/2]} |f_n(x)| = 1 < \infty$ (due to

the boundedness of functions fn(x)) and $\sup_{n \in N} \bigvee_{0}^{\pi/2} f_n = 1 < \infty$ (due to the monotonicity of functions

 $f_n(x)$ on the interval $[0, \pi/2]$, then, a subsequence converging at each point of $[0, \pi/2]$ to a certain function f with $\bigvee_{0}^{\pi/2} f \leq \lim_{n \to \infty} \bigvee_{0}^{\pi/2} f_n$ may be extracted from the sequence $\{f_n(x)\}$ based on Helly's theorem. The possibility of taking the initial function $f_0(x)$ as such function f will be shown below.

Theorem 1. A sequence of functions $f_n(x)$ converges to the initial function $f_0(x)$, with the convergence being point -by-point, though not uniform.

Proof. We have $f_n(x) - f_0(x) = 0$, $\forall n \in \mathbb{N}$ at x = 0 and $x = \pi/2$. Therefore, $f_n(x) \xrightarrow[n \to \infty]{} f_0(x)$ at these points, as $\forall \varepsilon > 0 \exists n^* \in \mathbb{N} \ \forall n : n > n^* \Rightarrow |f_n(x) - f_0(x)| < \varepsilon$. We may set $n^* = 1$ as an example.

 $\sin x > (2/\pi) \cdot x, \, \forall x \in (0, \pi/2),$ then the condition $f_n(x) = \sin((\pi/2) \cdot f_{n-1}(x)) > f_{n-1}(x) > \dots > f_1(x) > 0$ is fulfilled for any $x \in (0, \pi/2)$. Then, the sequence $f_n(x)$, $\forall x \in (0, \pi/2)$ is positive, ascending, and limited, and, therefore, it has the $\lim_{n \to \infty} f_n(x) = A \in \mathbf{R}$ finite limit, which will be indicated We obtain as $A = \lim_{n \to \infty} \sin((\pi/2) \cdot f_{n-1}(x)) = \sin((\pi/2) \cdot \lim_{n \to \infty} f_{n-1}(x)) = \sin((\pi/2) \cdot A), \text{ based on which we find}$ that A = 0 or A = 1. As the sequence is of positive terms and ascending, then $A = 1 = f_0(x)$. Then, $f_n(x) \xrightarrow[n \to \infty]{} f_0(x)$ on the considered interval. With the conclusion on convergence of the sequence at x = 0 and $x = \pi/2$, which was made above, we conclude that $f_n(x) \xrightarrow[n \to \infty]{} f_0(x), \forall x \in [0, \pi/2].$ This convergence is only a point-by-point one, but not uniform, as the function $f_0(x)$ is not continuous on the interval $[0, \pi/2]$.

Theorem 2. The sequence of approximating functions $f_n(x)$ converges along the norm towards the initial function $f_0(x)$ in Banach $L_1[0,\pi/2]$ and Hilbert spaces of measurable functions $L_2[0,\pi/2]$.

Proof. We introduce the sequence of functions $\{\eta_n(x) \mid \eta_n(x) = (2/\pi) \cdot \operatorname{arctg}(n\pi); n \in N\} \subset C^{\infty}[0, \pi/2]$, which are minorant with respect to the sequence $f_n(x)$. It may be shown that $f_n(x) \ge \eta_n(x), \forall n \in N, \forall x \in [0, \pi/2]$. It should be noted that the measure of the set of discontinuity points of the function $f_0(x)$ is zero.

Then, with functions $f_n(x)$ and $\eta_n(x)$ being non-negative terms and limited on the considered interval, we obtain the following in the space $L_1[0, \pi/2]$:

$$\left\|f_{0}(x) - f_{n}(x)\right\| = \int_{0}^{\pi/2} (1 - f_{n}(x))dx \le \int_{0}^{\pi/2} (1 - \eta_{n}(x))dx = \frac{\pi}{2} - \arctan\frac{\pi n}{2} + \frac{1}{\pi n} \cdot \ln\left(1 + (\pi n)^{2} / 4\right)$$

As
$$\lim_{n \to \infty} \left(\frac{\pi}{2} - \arctan\frac{\pi n}{2} + \frac{1}{\pi n} \cdot \ln\left(1 + (\pi n)^{2} / 4\right)\right) = 0, \text{ then } \left\|f_{0}(x) - f_{n}(x)\right\| \xrightarrow[n \to \infty]{} 0.$$

Similarly it may be proved that the sequence $f_n(x)$ converges along the norm towards the function $f_0(x)$ in the space $L_2[0, \pi/2]$.

Thus, the sequence of approximating functions $f_n(x)$ in spaces $L_1[-\pi,\pi]$ and $L_2[-\pi,\pi]$ is fundamental. Whereas, the sequence $f_n(x)$ is not fundamental in the space $C[-\pi,\pi]$.

The number $\pi/2$ was used in the sequence of approximating functions (3) as a constant factor; however, it is possible to take another factor, which may be variable as well. Cosine and

other trigonometric functions and their combinations may be used instead of sine in the suggested method of approximation. For example, if we use the sequence of recursive functions

$$\{f_n(x) \mid f_n(x) = \cos(\varphi_n(x)), \varphi_n(x) = (\pi/2) \cdot \sin(\varphi_{n-1}(x)), \varphi_1(x) = x, n-1 \in N\} \subset C^{\infty}[-\pi,\pi],$$

we may approximate short-term impulses. The graph of one function from such sequence is presented in Fig. 3.



Fig. 3. Graph of the analytical function that approximates short-term pulses

These functions may be used for mathematical models describing the transmission of short-term signals, mechanical systems with shock interactions, etc. It should be noted that, despite the impulse (highly nonlinear) shape of graphs of such functions, they are continuous analytical functions and tolerate the application of analytical methods. The error of the approximation in spaces $L_1[-\pi,\pi]$ and $L_2[-\pi,\pi]$ may be as small as we please in these case.

We return to the sequence of approximating functions (2). The function $f_1(x)$ will be called the initial function (or angle one). We may use another function (not necessarily a periodic one) instead of sine as the initial function. It should be noted that, when iterative procedure (2) is used and given condition $|f_1(x)| < 2$, we obtain $\lim_{n \to \infty} f_n(x) = \operatorname{sign}(f_1(x))$. In addition, we may approximate any step function. In fact, we will take the initial function written as $f_1(x) = \exp(1-(ax+b)^2) - 1$ for the step function

$$f(x) = \begin{cases} h, x \in (x_1, x_2), \\ 0, x \notin (x_1, x_2). \end{cases}$$
(3)

We obtain $a = 2/(x_1 - x_2)$; $b = (x_1 + x_2)/(x_2 - x_1)$ based on the condition $f_1(x_1) = f_1(x_2) = 0$. The sequence

 $\{f_n(x) \mid f_n(x) = (h/2) \cdot (1 + \sin \varphi_n(x)), \varphi_n(x) = (\pi/2) \cdot \sin \varphi_{n-1}, \varphi_1(x) = (\pi/2) \cdot f_1(x), n-1 \in N\}$ for these values of coefficients *a* and *b* converges to the step function f(x). Then, any step function with values $h_i i$ on intervals (x_{1i}, x_{2i}) may be approximated by the sum of identical

sequences $\sum_{i=1}^{k} \{ f_n(x) \}_i$.

When we considered approximation of the step function f(x) (3), we assumed that its position and height are precisely known. In actual problems parameters are usually set approximately. Let, for example, the initial parameters be set with absolute errors $|\hat{x}_1 - x_1| = \Delta x_1 \in [0, \Delta^* x_1), |\hat{x}_2 - x_2| = \Delta x_2 \in [0, \Delta^* x_2), |\hat{h} - h| = \Delta h \in [0, \Delta^* h),$ where $\Delta^* x_1 = \sup \Delta x_1, \Delta^* x_2 = \sup \Delta x_2, \Delta^* h = \sup \Delta h, \quad \hat{x}_1, \hat{x}_2, \hat{h} \text{ are approximated values of the$ parameters. We consider step function (3) on the interval <math>[c, d], for which $[x_1 - \Delta^* x_1, x_2 + \Delta^* x_2] \subset [c, d].$ In this case, we obtain the following estimated absolute errors of approximation with respect to the norm in spaces $L_1[c, d], L_2[c, d]$ and M[c, d]

respectively, with
$$M[c, d]$$
 being the set of functions with metric $\rho(f^{(1)}(x), f^{(2)}(x)) = \sup_{x \in [c,d]} \left| f^{(1)}(x) - f^{(2)}(x) \right|$ limited on the interval $[c,d]$:
 $\left\| \Delta f \right\| < \sup_{\Delta x_1 \ \Delta x_2 \ \Delta h} \lim_{n \to \infty} \left\| f(x) - f_n(x) \right\|_{L_1[c,d]} = \left(|h| + \Delta^* h \right) \cdot \left(\Delta^* x_1 + \Delta x_2 \right) + (x_2 - x_1) \cdot \Delta^* h;$
 $\left\| \Delta f \right\| < \sup_{\Delta x_1 \ \Delta x_2 \ \Delta h} \lim_{n \to \infty} \left\| f(x) - f_n(x) \right\|_{L_2[c,d]} = \sqrt{\left(|h| + \Delta^* h \right)^2 \cdot \left(\Delta^* x_1 + \Delta^* x_2 \right) + (x_2 - x_1) \cdot \Delta^* h^2};$
 $\left\| \Delta f \right\| < \sup_{\Delta x_1 \ \Delta x_2 \ \Delta h} \lim_{n \to \infty} \left\| f(x) - f_n(x) \right\|_{L_2[c,d]} = \sqrt{\left(|h| + \Delta^* h \right)^2 \cdot \left(\Delta^* x_1 + \Delta^* x_2 \right) + (x_2 - x_1) \cdot \Delta^* h^2};$

It can be seen from the obtained estimations that the error of approximation does not accumulate, which is a positive aspect of the suggested method.

As in practice we usually only know the approximate parameter values and measurement errors, it is more convenient to express the upper bound estimates for the absolute error of approximation as follows:

$$\begin{split} \left\| \Delta f \right\|_{L_{1}[c,d]} &< (\left| \hat{h} \right| + 2\Delta^{*}h) \cdot (\Delta^{*}x_{1} + \Delta^{*}x_{2}) + (\hat{x}_{2} - \hat{x}_{1} + \Delta^{*}x_{1} + \Delta^{*}x_{2}) \cdot \Delta^{*}h \,; \\ \left\| \Delta f \right\|_{L_{2}[c,d]} &< (\left| \hat{h} \right| + 2\Delta^{*}h)^{2} \cdot (\Delta^{*}x_{1} + \Delta^{*}x_{2}) + (\hat{x}_{2} - \hat{x}_{1} + \Delta^{*}x_{1} + \Delta^{*}x_{2}) \cdot \Delta^{*}h^{2} \,; \\ \left\| \Delta f \right\|_{\boldsymbol{M}[c,d]} &< \left| \hat{h} \right| + 2\Delta^{*}h \,. \end{split}$$

We return to (1) and its approximation using (2) in the space of limited functions $M[0, \pi]$.

Let $\Delta = |f_0(x) - f_n(x)| \in [0, 1]$ be the absolute error of approximation. We write down the sequence $\left\{ r_n \mid r_n = \max_{\Delta = x1, x2 \in [0, \pi]: f_n(x1) = f_n(x2)} |x2 - x1| \right\}$ of maximum metrics. Based on the equation $f_n(x) = 1 - \Delta$, we find that this sequence may be represented as $\left\{ r_n \mid r_n = \pi - 2 \arcsin \lambda_n, \lambda_n = (2/\pi) \cdot \arcsin \lambda_{n-1}, \lambda_1 = 1 - \Delta, n - 1 \in N \right\}$. Similarly to the proof of the theorem 1, it may be proved that $r_n(\Delta) \xrightarrow[n \to \infty]{} r^*(\Delta) = \begin{cases} \pi, \Delta \in (0, 1], \\ 0, \Delta = 0, \end{cases}$ with convergence on the interval [0, 1] being point-by -point without being uniform. It is

with convergence on the interval [0, 1] being point-by -point without being uniform. It is important to mention that the sequence $\{r_n\}$ converges to the step function as well.

The graphs of several of the initial functions from the sequence $r_n(\Delta)$ are presented in Fig. 4. It can be seen from Fig. 4 that the length of the interval on which the error of approximation does not exceed Δ rises sharply when n is increased in the area of rather small values of the error Δ . This proves the rapid convergence of the suggested method and is its positive peculiarity.

In order to quantitatively estimate the change in the length of this interval, we deduce the approximate dependence for the function $\Delta r(n, \Delta) = r_n - r_{n-1}$. To do this, we use the relationship $r_n - r_{n-1} = 2(x_{n-1} - x_n)$, where $x_n = \arcsin((2/\pi) \cdot x_{n-1})$, $x_1 = \arcsin(1-\Delta)$. Then we get $r_n - r_{n-1} = 2(x_{n-1} - \arcsin((2/\pi) \cdot x_{n-1}))$. When we expand $\arcsin((2/\pi) \cdot x_{n-1})$ into the Maclaurin series and consider that x_{n-1} values are rather small, we obtain approximately the following formula: $r_n - r_{n-1} \approx (2/\pi) \cdot (\pi - 2) \cdot x_{n-1}$. Then $r_n - r_{n-1} \approx (2/\pi)^{n-1} \cdot (\pi - 2) \cdot \arcsin(1-\Delta)$.



Fig. 4. Lengths of intervals with the error of approximation not exceeding Δ

We indicate several properties of the suggested approximation (2).

Property 1. The maximum difference in lengths of intervals $r_n - r_{n-1}$ does not depend on n and is obtained using the relationship $\max_{\Delta \in [0, 1]} (r_n - r_{n-1}) = \sqrt{\pi^2 - 4} - 2 \arcsin \sqrt{1 - 4/\pi^2}, n - 1 \in \mathbb{N}.$

Proof. Based on the previously obtained relationship $r_n - r_{n-1} = 2(x_{n-1} - \arcsin((2/\pi) \cdot x_{n-1})), n-1 \in \mathbb{N}$, we obtain the derivative $\frac{d(r_n - r_{n-1})}{d\Delta} = -2^{n-1} \cdot \left(\sqrt{\pi^2 - 4x_{n-1}^2} - 2\right) / \sqrt{\prod_{i=1}^{n-1} (\pi^2 - 4x_i^2)(1 - (1 - \Delta)^2)}$.

The points $x_{n-1} = x_{n-2} = \ldots = x_1 = \pi/2$ are minimum points, at which $r_n - r_{n-1} = 0$. We also obtain $r_n - r_{n-1} = 0$ in the case of $\Delta = 1$. The points $x_{n-1} = \sqrt{(\pi^2/4) - 1}$ are points and do Then, maximum not n_{\perp} depend on we obtain $\max_{r=10}^{n} (r_n - r_{n-1}) = \sqrt{\pi^2 - 4} - 2 \arcsin \sqrt{1 - 4/\pi^2}.$ It is indicated that $\Delta \in [0, 1]$

 $\max_{\Delta \in [0, 1]} (r_n - r_{n-1}) \approx 0,661 \text{ for use as a reference.}$

Property 2. The maximum difference in the values of functions $f_n(x) - f_{n-1}(x)$ does not depend on n and is obtained using the relationship $\max_{x \in [0, \pi]} (f_n(x) - f_{n-1}(x)) = (\sqrt{\pi^2 - 4} - 2\arccos(2/\pi))/\pi, n-1 \in \mathbb{N}.$

The proof is similar to the one for property 1. It is also indicated that $\max_{x \in [0, \pi]} (f_n(x) - f_{n-1}(x)) \approx 0.211$ for use as a reference.

Property 2 shows that the sequence of approximating functions $f_n(x)$ (2) is not Cauchy convergent; i.e., it is not fundamental, as $\exists \varepsilon > 0 \forall n^* \in N \exists n, m > n^*$, which is $\max_{x \in [0,\pi]} |f_n(x) - f_m(x)| > \varepsilon$. The number 0.1 may be taken as ε , for example, given that $m = n^* + 1, n = n^* + 2$.

The obtained relationships may be used to estimate errors of approximation in the solution of the applied problems.

3. EXAMPLES OF APPLICATION

The sequence of sine mechanisms may act as a mechanical analog of the suggested approximations [7]. For example, the sequence of sine mechanisms in a position that corresponds to approximation (3) is presented in Fig. 5. Here, (1) represents the drive shaft; (2) represents the crank; (3) is the slide block; (4) is the link; (5) represents the rack; and (6) is the gear wheel. Then, the sequence of elements is repeated (it may be repeated several times). Element 7 corresponds to the output element, which may be connected to the drive shaft through the pinion.



Fig. 5. Mechanical analog of the approximation of a step function

The mechanism presented in Fig. 5 transforms the uniform rotational motion of drive shaft *l* into the intermittent reciprocal or vibrational motion of the output element (with any degree of accuracy). In addition, the different relative positions of the cranks and the different relationships of the sizes of the crank and gear wheels make it possible to simulate various laws of motion of the output element that correspond to the considered approximations of the step functions (discontinuous motions, impulsive motions, etc.). Such a mechanism may be applied, for example, as transport mechanism in tape drive systems to provide a higher quality of the execution of the process. It is also possible to apply such a mechanism in pulse variators to achieve a more uniform motion of the output drive shaft, as the vibrational processes occur in accordance with the curves composed of the segments being approximated to a constant, rather than with a sine wave.

The suggested method of approximation makes it possible to obtain functions which may be applied, for example, in the design and manufacturing of gear wheels and spline joints. The result of the construction of gear profile in MathCAD software is presented in Fig. 6. The function $r(\varphi) = A + a \cdot \sin((\pi/2) \cdot \sin(n\varphi))$, where A is the pitch circle radius, a is the tooth point height, n is the number of teeth, and r and φ are polar radius and polar angle, respectively, was taken as a basis for the construction of the profile.

When we change the number of nested trigonometric functions used for approximation and vary their parameters, we may get gear profiles and spline joints with enhanced reliability in comparison with evolvent ones, which, in contrast with rectangular splines, for example, have no significant stress concentrators.

We consider one more example of the application of the developed methods of approximation. The optical installation for sound reconstruction from worn and damaged

records, which makes it possible to obtain electronic profiles of sound carrying media via a noncontact method, was developed in the United States (Figs. 7a, 7b) [8].

Several tens (sometimes even hundreds) of thousands of electronic radial profiles of each sound carrying medium being reconstructed are acquired using the optical installation. Approximation, mostly the one using linear splines (Fig. 7c), is used for the reconstruction of electronic profiles. The accuracy of the approximation leaves much to be desired in this case, whereas the electronic analog of the initial profile may be reconstructed with a rather high degree of accuracy using the methods suggested in the present paper (Fig. 7e).



Fig.6. Construction of a gear profile using the approximation of step functions

The black dots in Fig. 7d indicate the measurements of the radial profile of sound tracks using the optical installation. An enlarged image of Fig. 7d is presented in Fig. 7e, where the white line corresponds to an approximation of the profile using the suggested technique.



Fig. 7. Profiles of sound tracks and their application

The described methods of approximation make it possible to automate the process of the reconstruction of electronic sound tracks, which is very important for the execution of the process with a high performance level, taking into consideration the significant number (tens and hundreds of thousands) of electronic radial profiles.

The suggested methods of approximation may also be used in the mathematical modeling of biomedical processes. For example, a fragment of a cardiogram is presented in Fig. 8. Approximation is carried out for one of the graphs of the cardiogram using the proposed procedure. The graphic results of the approximation by means of one of the developed functions are presented in an enlarged form in the middle part of the figure (Fig. 8a), where the approximating function is superimposed on the graph of the cardiogram. In order to better understand the graph of the approximating function, this graph in Fig. 8b is presented in a position shifted with respect to the cardiogram. It can be seen that the approximation is quite accurate. Similar approximations may be carried out for other graphs of the cardiogram as well.



Fig. 8. Approximation of a fragment of the cardiogram

The possibility of using the proposed methods for approximation of nonperiodic step functions should also be noted. The period of approximating functions in this case should be rather large covering the area of possible values of the argument of the function being approximated of the actual process being investigated. Such an approximation may be used in the modeling of, for example, technical systems with dry friction parameters and inertial transformer dynamics with an alternating -sign moment of resistance.

4. NUMERICAL TESTING

Numerical testing of the proposed approximating procedure will be carried out using the example of investigation of dynamics of an inertial impulse stepless gear. It is known [9] that weak elements (free wheel mechanisms) may be excluded from the construction of the inertial

impulse steplsess gear based on planetary gear with unbalanced satellites under condition that the moment of resistance affecting the drive shaft has an alternating sign. The gear's dynamics may be described by the highly nonlinear second-order differential equation

$$A_1 \beta + A_2 (\omega - \beta)^2 - A_3 \omega^2 = -M_C,$$

where

 $A_1 = B_1 + b_1 \cdot \cos \psi, A_2 = a_2 \cdot \sin \psi, A_3 = a_3 \cdot \sin \psi, \psi = q(\omega \cdot t - \beta),$

 B_1, b_1, a_2, a_3, q are constant coefficients, including the gear parameters,

 $M_C = M_1 \cdot \text{sign}(\beta) + M_0$ is the moment of resistance affecting the drive shaft $(M_0, M_1 \equiv const),$

 $\omega \equiv const$ is the angular velocity of the driveshaft,

 β is the rotation angle of the driven shaft, and $\bullet = \frac{d}{dt}$ is the operator of derivation with respect

to time t.

The sign function sign(β) is highly nonlinear, which complicates carrying out analytical investigations of the dynamics of the inertial impulse gear. In addition, this function is not periodic. We approximate the sign function using the suggested methods (2) by, for example, an analytical function written as $sign(\beta) \approx f_4(\beta/10)$. It should be noted that we take relatively small n = 4 for the approximation, leaving substantial opportunities for a reduction in the approximation error.

For the sake of comparison, we carry out a numerical solution of the differential motion equation with the sign and the approximating functions for particular examples of gears

according to the Runge–Kutta method. Phase trajectories on phase plane (β, β) with access to a periodic solution are presented in Fig. 9. Here, the solid line indicates the solution obtained with the gear with a discontinuous sign function used in the mathematical model, while the dotted line represents the solution obtained using an analytical approximation. The thickened line in Fig. 9 corresponds to the periodic solution.



Fig. 9. Phase trajectories in the case of use of the sign function and its approximation

It can be seen from the figure that the error of the results is not large, which shows good convergence of the suggested approximating procedures. Furthermore, the approximation error may be reduced to as small value as desired through an increase in the number of nested functions.

The considered examples are taken from various areas and are not single ones for the application of the suggested methods of approximation. Therefore, sufficient universality of these methods may be stated.

The described methods of approximation do not have any of the disadvantages of expansions of functions into Fourier series and may find wide use in the solution of applied problems. It should also be noted that the proposed approximating functions are continuous and analytical ones. They reflect actual processes to a larger extent than step functions, as even jump processes occur in reality within short, but not zero, time intervals.

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