Solving Fredholm integral equations with application of the four Chebyshev polynomials

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Abstract

In this work, we study the approximation of the Fredholm integral equation of the second kind using the four Chebyshev series expansions. Those equations will be solved using m collocation points. That is to say, we will make the residual equal to zero at m points, giving us a system of m linear equations.

Key words Chebyshev polynomials, Fredholm integral equation, Collocation method, Numerical method.

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1. Introduction

Some phenomena which appear in many areas of scientific fields such as plasma physics, fluid dynamics, mathematical biology and chemical kinetics can be modelled by Fredholm integral equations [6]. Also this type of equations occur of scattering and radiation of surface water wave, where we can transform any ordinary differential equation of the second order with boundary conditions into a Fredholm integral equation.

$$\varphi(t_0) - \int_{-1}^{1} k(t,t_0)\varphi(t)dt = f(t_0), \tag{1}$$

with a given kernel $k(t,t_0)$ and a function $f(t)$, we try to find the unknown function $\varphi(t)$ as in [3,7] where the authors estimate the density function $\varphi(t)$ by means of Legendre and the first Chebyshev polynomials. For this study we replace the function $\varphi(t)$ by the four Chebyshev polynomials and compare the accuracy of the estimation of the unknown function with many numerical examples.

2. Discretization of integral equation

In this section, we apply a collocation method to the equation(1) in order to discredit and convert it to a system of linear equations. For this latter by using a Chebyshev polynomials we approximate the unknown $\varphi(t)$ such that

$$\varphi(t) = \varphi(t) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n S_n(x), \tag{2}$$

where $S_n(x)$ denotes the nth Chebyshev polynomial of the first, second, third or fourth kind. So, that these series behave like Fourier series. Thus in particular, this series converges pointwise to φ on $[-1,1]$ if φ is continuous there, while the convergence is uniform if φ satisfies a Dini-Lipschitz condition or is of bounded variation, see, e.g. [1,2]. Then truncations of the series provide polynomials with good approximation properties on the interval $[-1,1]$..

The four Chebyshev polynomials with the interval of orthogonality $[-1,1]$, see [2,4] are defined as

1- **The first-kind polynomial** T_n

$$T_n(x) = \cos n\theta \quad \text{when} \quad x = \cos\theta$$

The three term recurrence formula satisfied by Chebyshev polynomials is the translation of the elementary trigonometric identity

$$\cos n\theta + \cos(n-2)\theta = 2\cos\theta\cos(n-1)\theta,$$

which becomes

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x), \quad n = 2, 3, ...$$

With

$$T_0(x) = 1, \quad T_1(x) = x$$

2- **The second-kind polynomial** U_n

$$U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta} \quad \text{when} \quad x = \cos\theta$$

The three term recurrence formula satisfied by Chebyshev polynomials is the translation of the elementary trigonometric identity

$$\sin(n+1)\theta + \sin(n-1)\theta = 2\cos\theta\sin n\theta,$$

which becomes

$$U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x), \quad n = 2, 3, ...$$

With

$$U_0(x) = 1, \quad U_1(x) = 2x$$

3- **The third-kind polynomial** U_n

$$V_n(x) = \frac{\cos(n+\frac{1}{2})\theta}{\cos\frac{1}{2}\theta} \quad \text{when} \quad x = \cos\theta$$

The three term recurrence formula satisfied by Chebyshev polynomials is the translation of the elementary trigonometric identity

$$\cos(n+\frac{1}{2})\theta + \cos(n-2+\frac{1}{2})\theta = 2\cos\theta\cos(n-1+\frac{1}{2})\theta,$$

which becomes

$$V_n(x) = 2xV_{n-1}(x) - V_{n-2}(x), \quad n = 2, 3, ...$$

With

$$V_0(x) = 1, \quad V_1(x) = 2x - 1$$

4- **The fourth-kind polynomial** W_n

$$W_n(x) = \frac{\sin(n+\frac{1}{2})\theta}{\sin\frac{1}{2}\theta} \quad \text{when} \quad x = \cos\theta$$

The three term recurrence formula satisfied by Chebyshev polynomials is the translation of the elementary trigonometric identity

$$\sin(n+\frac{1}{2})\theta + \sin(n-2+\frac{1}{2})\theta = 2\cos\theta\sin(n-1+\frac{1}{2})\theta,$$

which becomes

$$W_n(x) = 2xW_{n-1}(x) - W_{n-2}(x), \quad n = 2, 3, ...$$

With

$$V_0(x) = 1, \quad V_1(x) = 2x + 1.$$

By substituting the relation (2) in the equation(1) we get

$$\sum_{n=0}^{m} c_n S_n(t_0) - \int_{-1}^{1} k(t,t_0) \sum_{n=0}^{m} c_n S_n(t_0) = f(t_0). \tag{3}$$

Choosing the equidistant collocation points as follows

$$t_j = -1 + \frac{2j}{m}, \quad j = 0, 1, ...m, \tag{4}$$

and define the residual as

$$R_n(t_0) = \sum_{n=0}^{m} c_n S_n(t_0) - \int_{-1}^{1} k(t,t_0) \sum_{n=0}^{m} c_n S_n(t) - f(t_0)$$

Then, by imposing conditions at collocation points

$$R_n(t_j) = 0, \quad j = 0, 2, ...m,$$

the integral equation (3) is converted to a system of linear equations.

Theorem

Let $A : X \to X$ be compact and the equation

$$(I - A)\varphi = f, \tag{5}$$

admit a unique solution.Assume that the projections $P_n : X \to X_n$ satisfy to $\|P_n A - A\| \to 0,\ n \to \infty$. Then, for sufficiently large n, the approximate equation

$$\varphi_n - P_n A \varphi_n * P_n f, \tag{6}$$

has a unique solution for all $f \in X$ and there holds an error estimate

$$\|\varphi - \varphi_n\| \leq M \|\varphi - P_n\varphi\|, \tag{7}$$

with some positive constant M depending on A.

Proof

As it is known for all sufficiently large n the inverse operators $(I - P_nA)^{-1}$ exist and are uniformly bounded, see [1,5]. To verify the error bound, we apply the projection operator P_n to the equation (5) and get

$$P_n\varphi - P_nA\varphi = P_nf,$$

or again

$$\varphi - P_nA\varphi = P_nf + \varphi - P_n\varphi.$$

Subtracting this from (6) we find

$$(I - P_nA)(\varphi - \varphi_n) = (I - P_n)\varphi.$$

Hence the estimate (7) follows.

3. Numerical examples

Example 1

Consider the Fredholm integral equation

$$\varphi(t_0) - \int_{-1}^{1} \exp(2t_0 - \frac{5}{3}t)\varphi(t)dt = \exp(2t_0)(1 - 3\exp(\frac{1}{3}) + 3\exp(-\frac{1}{3}),$$

where the function $f(t_0)$ is chosen so that the solution $\varphi(t)$ is given by

$$\varphi(t) = \exp(2t)$$

The approximate solution $\widetilde{\varphi}(t)$ of $\varphi(t)$ is obtained by the solution of the system of linear equations for $N = 10$

Points of t	Exact solution	Approx solution	Error
-1.0000	1.353353e-001	1.353353e-001	2.915452e-008
-0.8000	2.018965e-001	2.018965e-001	4.349344e-008
-0.6000	3.011942e-001	3.011941e-001	6.488458e-008
-0.4000	4.493290e-001	4.493289e-001	9.679642e-008
-0.2000	6.703200e-001	6.703199e-001	1.444033e-007
0.0000	1.000000e+000	9.999998e-001	2.154244e-007
0.2000	1.491825e+000	1.491824e+000	3.213754e-007
0.4000	2.225541e+000	2.225540e+000	4.794358e-007
0.8000	3.320117e+000	3.320116e+000	7.152342e-007
0.8000	4.953032e+000	4.953031e+000	1.067004e-006
1.0000	7.389056e+000	7.389055e+000	1.591783e-006

Table 1. The exact and approximate solutions of example 1 in some arbitrary points, of the system of linear equations

Example 2
Consider the Fredholm integral equation

$$\varphi(t_0) - \int_0^1 (t_0 - t)^3 \varphi(t)dt = \frac{1}{1+2t_0^2} - \frac{t_0}{2}(6 + \sqrt{2}(-3 + 2t_0^2)\arctan(\sqrt{2})),$$

where the function $f(t_0)$ is chosen so that the solution $\varphi(t)$ is given by

$$\varphi(t) = \frac{1}{1+2t^2}.$$

The approximate solution $\widetilde{\varphi}(t)$ of $\varphi(t)$ is obtained by the solution of the system of linear equations for $N = 10$

Points of t	Exact solution	Approx solution	Error
-1.0000	3.333333e-001	3.333334e-001	2.597979e-008
-0.8000	4.385965e-001	4.385966e-001	1.046103e-007
-0.6000	5.813953e-001	5.813955e-001	1.251738e-007
-0.4000	7.575758e-001	7.575759e-001	1.043146e-007
-0.2000	9.259259e-001	9.259260e-001	5.867663e-008
0.0000	1.000000e+000	1.000000e+000	4.904144e-009
0.2000	9.259259e-001	9.259259e-001	4.035875e-008
0.4000	7.575758e-001	7.575757e-001	6.046791e-008
0.8000	5.813953e-001	5.813953e-001	3.877922e-008
0.8000	4.385965e-001	4.385965e-001	4.135147e-008
1.0000	3.333333e-001	3.333335e-001	1.965683e-007

Table 2. The exact and approximate solutions of example 2 in some arbitrary points, of the system of linear equations

4. Conclusion

In this work, a projection method known as collocation method with the four Chebyshev polynomials was chosen to discretize the Fredholm integral equations. This method has some advantages, there is no difference between the four Chebyshev polynomials. It is easy to require best convergence and less computations than other methods discussed in [3, 7]. In some methods the kernels of equations and the second member are required to satisfy some conditions.

5. References

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