Resolution of Nonlinear Partial Differential Equations by Elzaki Transform Decomposition Method

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November 23, 2015

Abstract

The aim of this work is to extend the application of Elzaki transform decomposition method suggested by M. Khalid et al. to resolve nonlinear partial differential equations. We apply the proposed method to obtain approximate analytical solutions of the proposed problems. Comparison between the numerical and the exact solutions revealed that (ETDM) is an alternative analytical method for solving nonlinear partial differential equations.

Keywords: Adomian decomposition method , Elzaki transform method, nonlinear partial differential equations.
2010MSC: 35A08; 44A05; 35G25.

1 Introduction

There is no secret to the researcher in the field of nonlinear partial differential equations, that the solution of this class of equations is not easy. So we find that many researchers have done and are still doing great efforts to find methods to solve this type of equations. These efforts resulted in the consolidation of this research field in many methods, among them we find the Adomian decomposition method (ADM) ([1]-[5]), variational iteration method (VIM) ([6]-[10]) and homotopy perturbation method (HPM) ([11]-[15]), which have become known in a large number of researchers in this area. A new option emerged recently, includes the composition of Laplace transform, Sumudu transform or Elzaki transform with these methods. Among wich are the Adomian decomposition method coupled with Laplace transform method ([16], [17]), Adomian decomposition Sumudu transform method [18], variational iteration method coupled with Laplace transform method ([19], [20]), variational iteration Sumudu transform method [21], homotopy perturbation transform method [22], homotopy perturbation Sumudu transform method ([23]-[25]), homotopy perturbation Elzaki transform method [27], Elzaki transform decomposition algorithm [29].

The basic motivation of the present study is to extend the application of the Elzaki transform decomposition algorithm suggested in [29] to solve nonlinear partial differential equations. The advantage of this method is its capability of combining two powerful methods for obtaining exact solutions for nonlinear equations. Several examples are given to re-confirm the effectiveness of this method.

2 Basic definitions of ELzaki Transform

A new integral transform called ELzaki transform ([26]-[28]) defined for functions of exponential order, is proclaimed. We consider functions in the set A defined by,

$$A = \left\{ f(t)/M,\ k_1, k_2 > 0,\ |f(t)| < Me^{\frac{|t|}{k_j}},\ if\ t \in (-1)^j \times [0,\ \infty) \right\}.$$

Definition 1 *If $f(t)$ is function defined for all $t \geqslant 0$, its Elzaki transform is the integral of $f(t)$ times $e^{-\frac{t}{s}}$ from $t = 0$ to ∞. It is a function of s and is defined by $E[f]$*

$$E\left[f(t)\right] = T(s) = s\int_0^\infty f(t)e^{-\frac{t}{s}}dt.$$

Theorem 2 *ELzaki transform amplifies the coefficients of the power series function,*

$$f(t) = \sum_{n=0}^\infty a_n t^n. \tag{1}$$

On the new integral transform "ELzaki Transform"
Is

$$E\left[f(t)\right] = T(v) = \sum_{n=0}^{\infty} n! a_n v^{n+2}. \tag{2}$$

Theorem 3 *Let $f(t)$ be in A and Let $T_n(v)$ denote ELzaki transform of nth derivative, $f^{(n)}(t)$ of $f(t)$, then for $n \geq 1$,*

$$T_n(v) = \frac{T(v)}{v^n} - \sum_{k=0}^{n-1} v^{2-n+k} f^{(k)}(0). \tag{3}$$

To obtain Elzaki transform of partial derivative we use integration by parts, and then we have

$$\begin{aligned} E\left(\frac{\partial f(x,t)}{\partial t}\right) &= \frac{1}{v}T(x,v) - vf(x,0), \\ E\left(\frac{\partial^2 f(x,t)}{\partial t^2}\right) &= \frac{1}{v^2}T(x,v) - f(x,0) - v\frac{\partial f(x,0)}{\partial t}, \end{aligned} \tag{4}$$

Properties of Elzaki transform can be found in Refs. ([26], [27]), we mention only the following

1. $E(1) = v^2$;
2. $E(t) = v^3$;
3. $E(t^n) = n! v^{n+2}$;
4. $E^{-1}(v^{n+2}) = \frac{t^n}{n!}$.

3 Elzaki Transform Decomposition Method for PDEs

In this section, we extend the proposed method [29] to solve ordinary differential equations, a new modified method for solving partial differential equations. To illustrate the basic idea of this method, we consider a general nonlinear nonhomogeneous partial differential equation

$$\frac{\partial^m u(x,t)}{\partial t^m} + Ru(x,t) + Nu(x,t) = g(x,t), \tag{5}$$

where $m = 1, 2, 3$, with the initial conditions

$$\left.\frac{\partial^{m-1} u(x,t)}{\partial t^{m-1}}\right|_{t=0} = f_{m-1}(x), \qquad m = 1, 2, 3, \tag{6}$$

where $\frac{\partial^m u(x,t)}{\partial t^m}$ is the partial derivative of the function $u(x,t)$ of order m ($m = 1, 2, 3$), R is the linear differential operator, N represents the general nonlinear differential operator, and $g(x,t)$ is the source term.

Applying the Elzaki Transform (denoted in this paper by E) on both sides of Eq. (5), we get

$$E\left[\frac{\partial^m u(x,t)}{\partial t^m}\right] + E\left[Ru(x,t)\right] + E\left[Nu(x,t)\right] = E\left[g(x,t)\right]. \tag{7}$$

Using the properties of Elzaki Transform, we obtain

$$v^{-m}E\left[u(x,t)\right] = \sum_{k=0}^{m-1} v^{2-m+k}\frac{\partial^k u(x,0)}{\partial t^k} + E\left[g(x,t)\right] - E\left[Ru(x,t) + Nu(x,t)\right], \tag{8}$$

where $m = 1, 2, 3$.

And thus, we have

$$E\left[u(x,t)\right] = \sum_{k=0}^{m-1} v^{2+k}\frac{\partial^k u(x,0)}{\partial t^k} + v^m E\left[g(x,t)\right] - v^m E\left[Ru(x,t) + Nu(x,t)\right]. \tag{9}$$

Operating the inverse transform on both sides of Eq. (9), we get

$$u(x,t) = G(x,t) - E^{-1}\left(v^m E\left[Ru(x,t) + Nu(x,t)\right]\right), \tag{10}$$

where $G(x,t)$, represents the term arising from the source term and the prescribed initial conditions.

The second step in Elzaki Transform Decomposition Method, is that we represent the solution as an infinite series given below

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t), \tag{11}$$

and the nonlinear term can be decomposed as

$$Nu(x,t) = \sum_{n=0}^{\infty} A_n, \tag{12}$$

where A_n are Adomian polynomials [30] of $u_0, u_1, u_2, ..., u_n$ and it can be calculated by the formula given below

$$A_n = \frac{1}{n!}\frac{\partial^n}{\partial\lambda^n}\left[N\left(\sum_{i=0}^{\infty}\lambda^i u_i\right)\right]_{\lambda=0}, \quad n = 0, 1, 2, \cdots. \tag{13}$$

Substituting (11) and (12) in (10), we have

$$\sum_{n=0}^{\infty} u_n = G(x,t) - E^{-1}\left[u^m E\left[R\sum_{n=0}^{\infty} u_n + \sum_{n=0}^{\infty} A_n\right]\right]. \tag{14}$$

On comparing both sides of the Eq. (14), we get

$$\begin{aligned} u_0(x,t) &= G(x,t), \\ u_1(x,t) &= -E^{-1}\left[u^m E\left[Ru_0(x,t) + A_0\right]\right], \\ u_2(x,t) &= -E^{-1}\left[u^m E\left[Ru_1(x,t) + A_1\right]\right], \\ u_3(x,t) &= -E^{-1}\left[u^m E\left[Ru_2(x,t) + A_2\right]\right], \\ &\vdots \end{aligned} \tag{15}$$

In general, the recursive relation is given as

$$u_{n+1}(x,t) = -E^{-1}\left[u^m E\left[Ru_n(x,t) + A_n\right]\right], \tag{16}$$

where $m = 1, 2, 3,$ and $n \geqslant 0$.

Finally, we approximate the analytical solution $u(x,t)$ by truncated series

$$u(x,t) = \lim_{N\to\infty}\sum_{n=0}^{N} u_n(x,t). \tag{17}$$

The above series solutions generally converge very rapidly [31].

4 Application of the ETDM

In this section, we apply Elzaki transform decomposition method for PDEs to solve nonlinear partial differential equations of the first, second and third order.

Example 4.1

First, we consider the following nonlinear partial differential equation

$$u_t + uu_x - u_{xx} = 0, \tag{18}$$

with initial condition

$$u(x,0) = x. \tag{19}$$

Applying the Elzaki transform on both sides of Eq. (18). Thus, we get

$$E\left[\ u_t\right] + E\left[uu_x\right] - E\left[u_{xx}\right] = 0. \tag{20}$$

We use the properties of Elzaki transform, we have

$$E\left[\ u(x,t)\right] = xv^2 - vE\left[uu_x - u_{xx}\right]. \tag{21}$$

Taking the inverse Elzaki transform on both sides of Eq. (21), we obtain

$$u(x,t) = x - E^{-1}[vE\left[uu_x - u_{xx}\right]]. \tag{22}$$

By applying the aforesaid decomposition method, we have

$$\sum_{n=0}^{\infty} u_n(x,t) = x - E^{-1}\left[vE\left[\sum_{n=0}^{\infty} A_n(u) - \sum_{n=0}^{\infty} (u_n)_{xx}\right]\right] \tag{23}$$

On comparing both sides of Eq. (23), we get

$$\begin{aligned} u_0(x,t) &= x, \\ u_1(x,t) &= -E^{-1}\left[vE\left[A_0(u) - u_{0xx}(x,t)\right]\right], \\ u_2(x,t) &= -E^{-1}\left[vE\left[A_1(u) - u_{1xx}(x,t)\right]\right], \\ u_3(x,t) &= -E^{-1}\left[vE\left[A_2(u) - u_{2xx}(x,t)\right]\right], \\ &\vdots \end{aligned} \tag{24}$$

The first few components of $A_n(u)$ polynomials [30], for example, are given by

$$\begin{aligned} A_0(u) &= u_0 u_{0x}, \\ A_1(u) &= u_0 u_{1x} + u_1 u_{0x}, \\ A_2(u) &= u_0 u_{2x} + u_2 u_{0x} + u_1 u_{1x}, \\ &\vdots \end{aligned} \tag{25}$$

Using He's polynomials (25) and the iteration formulas (24) we obtain

$$\begin{aligned} u_0(x,t) &= x, \\ u_1(x,t) &= -xt, \\ u_2(x,t) &= xt^2, \\ u_3(x,t) &= -xt^3, \\ &\vdots \end{aligned} \tag{26}$$

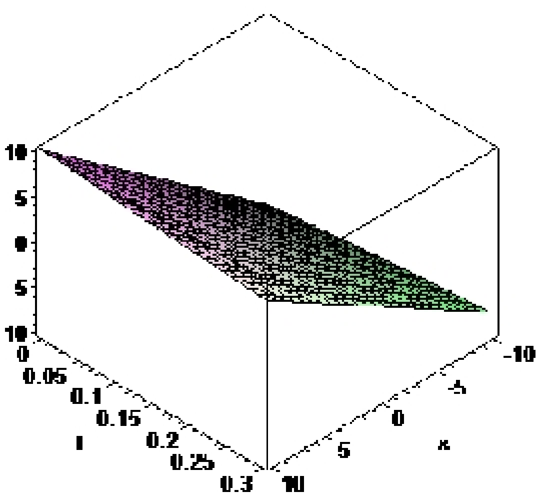
The first four terms of the decomposition series solution for Eq. (18) is given by

$$u(x,t) = x - xt + xt^2 - xt^3 + \cdots \tag{27}$$

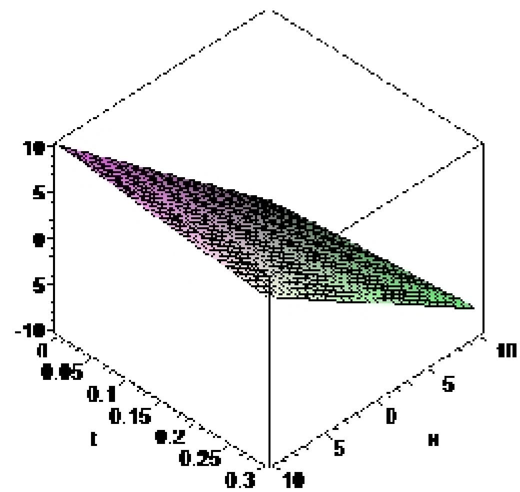
That gives

$$u(x,t) = \frac{x}{1+t}, \quad |t| < 1, \tag{28}$$

which is an exact solution to the KdV equation as presented in [32].



Exact solution (28) of Eq. (18-19).



Approximation solution of Eq. (18-19) by ETDM.

Example 4.2

Next, we consider a nonlinear partial differential equation of second order

$$u_{tt} - 2\frac{x^2}{t} u u_x = 0, \; t > 0, \tag{29}$$

with the initial conditions

$$u(x,0) = 0, \quad u_t(x,0) = x. \tag{30}$$

The exact solution of this equation is given by

$$u(x, y) = \tan(xt). \tag{31}$$

Applying the Elzaki transform and its inverse on both sides of Eq. (29) gives the following

$$u(x, t) = xt + 2E^{-1}[v^2 E\left[\frac{x^2}{t} u u_x\right]]. \tag{32}$$

By applying the aforesaid decomposition method, we have

$$\sum_{n=0}^{\infty} u_n(x, y) = xt + 2E^{-1}\left[v^2 E\left[\frac{x^2}{t}\sum_{n=0}^{\infty} A_n(u)\right]\right]. \tag{33}$$

On comparing both sides of Eq. (33), we get

$$\begin{aligned} u_0(x, t) &= xt, \\ u_1(x, t) &= 2E^{-1}\left[v^2 E\left[\tfrac{x^2}{t} A_0(u)\right]\right], \\ u_2(x, t) &= 2E^{-1}\left[v^2 E\left[\tfrac{x^2}{t} A_1(u)\right]\right], \\ u_3(x, t) &= 2E^{-1}\left[v^2 E\left[\tfrac{x^2}{t} A_2(u)\right]\right], \\ &\vdots \end{aligned} \tag{34}$$

Using the iteration formulas (34) and He's polynomials (25), we obtain

$$\begin{aligned} u_0(x, t) &= xt, \\ u_1(x, t) &= \tfrac{1}{3}x^3t^3, \\ u_2(x, t) &= \tfrac{2}{15}x^5t^5, \\ u_3(x, t) &= \tfrac{17}{315}x^7t^7. \end{aligned} \tag{35}$$

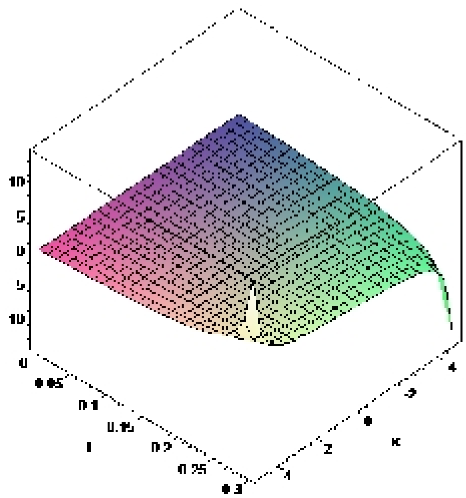
The approximate solution in a serie form is given by

$$u(x, t) = xt + \frac{1}{3}(xt)^3 + \frac{2}{15}(xt)^5 + \frac{17}{315}(xt)^7 + \circ (xt)^8. \tag{36}$$

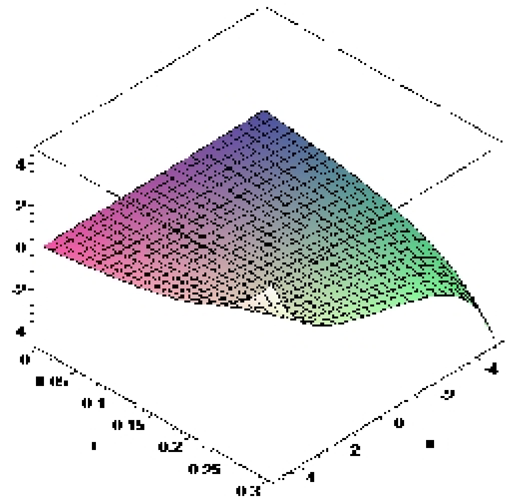
And so, we get the exact solution of Eq. (29) that is given in the form

$$u(x, t) = \tan(xt), \tag{37}$$

which is an exact solution to the nonlinear partial differential equation (29) of second order.



Exact solution (31) of Eq. (29-30).



Approximation solution of Eq. (29-30) by ETDM.

Example 4.3

Finaly, we consider a nonlinear partial differential equation of third order

$$u_{ttt} - 3uu_{xx} = 0,\ t > 0, \tag{38}$$

with the initial conditions

$$u(x,0) = \frac{1}{x},\ u_t(x,0) = \frac{1}{x^2},\ u_{tt}(x,0) = \frac{2}{x^3}. \tag{39}$$

The exact solution of the equation (38), is given by

$$u(x,y) = \frac{1}{x-t}, \quad \left|\frac{t}{x}\right| < 1,\ x \neq 0. \tag{40}$$

Applying the Elzaki transform and its inverse on both sides of Eq. (38) gives the following

$$u(x,t) = \frac{1}{x} + \frac{t}{x^2} + \frac{t^2}{x^3} + 3E^{-1}[v^3 E\,[uu_{xx}]]. \tag{41}$$

By applying the aforesaid decomposition method, we have

$$\sum_{n=0}^{\infty} u_n = \frac{1}{x} + \frac{t}{x^2} + \frac{t^2}{x^3} + 3E^{-1}\left[u^3 E\left[\sum_{n=0}^{\infty} A_n(u)\right]\right]. \tag{42}$$

On comparing both sides of Eq. (42), we get

$$\begin{aligned} u_0(x,t) &= \tfrac{1}{x} + \tfrac{t}{x^2} + \tfrac{t^2}{x^3}, \\ u_1(x,t) &= 3E^{-1}\left[v^3 E\left[A_0(u)\right]\right], \\ u_2(x,t) &= 3E^{-1}\left[v^3 E\left[A_1(u)\right]\right], \\ u_3(x,t) &= 3E^{-1}\left[v^3 E\left[A_2(u)\right]\right], \\ &\vdots \end{aligned} \tag{43}$$

The first few components of $A_n(u)$ polynomials [30], for example, are given by

$$\begin{aligned} A_0(u) &= u_0 u_{0xx}, \\ A_1(u) &= u_0 u_{1xx} + u_1 u_{0xx}, \\ A_2(u) &= u_0 u_{2xx} + u_1 u_{1xx} + u_2 u_{0xx}, \\ &\vdots \end{aligned} \tag{44}$$

Using the iteration formulas (43) and He's polynomials (44), we obtain

$$\begin{aligned} u_0(x,t) &= \tfrac{1}{x}\left[1 + \tfrac{t}{x} + \tfrac{t^2}{x^2}\right], \\ u_1(x,t) &= \tfrac{1}{x}\left[\tfrac{t^3}{x^3} + \tfrac{t^4}{x^4} + \tfrac{t^5}{x^5} + \tfrac{9}{20}\tfrac{t^6}{x^6} + \tfrac{6}{35}\tfrac{t^7}{x^7}\right], \\ u_2(x,t) &= \tfrac{1}{x}\left[\tfrac{11}{20}\tfrac{t^6}{x^6} + \tfrac{29}{35}\tfrac{t^7}{x^7} + \tfrac{t^8}{x^8} + \tfrac{15}{28}\tfrac{t^9}{x^9} + \tfrac{3}{20}\tfrac{t^{10}}{x^{10}}\right], \\ &\vdots \end{aligned} \tag{45}$$

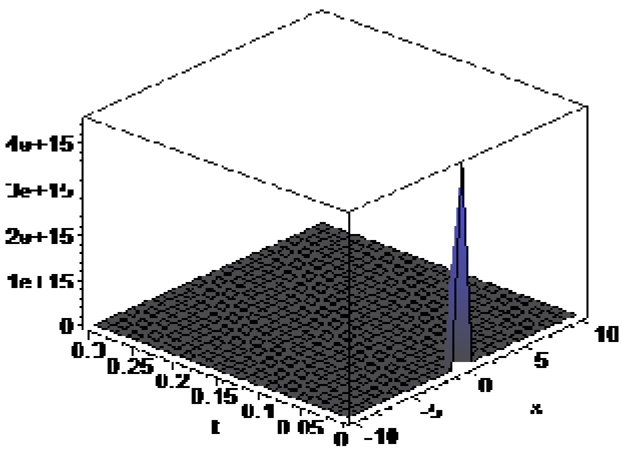
The first terms of the approximate solution of Eq. (38), is given by

$$\begin{aligned} U(x,t) \;=\; & \frac{1}{x}\left[1 + \left(\frac{t}{x}\right)^1 + \left(\frac{t}{x}\right)^2 + \left(\frac{t}{x}\right)^3 + \left(\frac{t}{x}\right)^4 + \left(\frac{t}{x}\right)^5 + \left(\frac{t}{x}\right)^6\right. \\ & \left. + \left(\frac{t}{x}\right)^7 + \left(\frac{t}{x}\right)^8 + \frac{15}{28}\left(\frac{t}{x}\right)^9 + \frac{3}{20}\left(\frac{t}{x}\right)^{10} + \cdots\right]. \end{aligned} \tag{46}$$

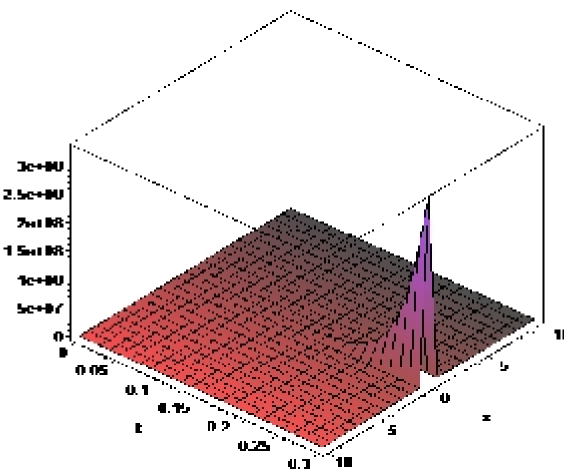
That gives

$$U(x,t) = \frac{1}{x-t}, \quad \left|\frac{t}{x}\right| < 1, \;\; x \neq 0, \tag{47}$$

which is an exact solution to the nonlinear partial differential equation (38) of third order.



Exact solution (40) of Eq. (38-39).



Approximation solution of Eq. (38-39) by ETDM.

5 Conclusion

The coupling of Adomian decomposition method (ADM) and Elzaki transform method proved very effective to solve nonlinear partial differential equations. The modified algorithm is suitable for such problems and is very userfriendly. From the obtained results, it is clear that the Elzaki transform decomposition method yields very accurate approximate solutions using only a few iterates, especially in the equations of the third order, where we note the emergence of the decomposition series solution after calculating the two first terms only. As a result, the conclusion that comes through this work is that (ETDM) can be applied to other nonlinear partial differential equations of higher order, due to the efficiency and flexibility in the application to get the possible results.

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