Approximation of Non $L^2(R)$ Functions on a Compact Interval with a Wavelet Base

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Abstract

Comparing the approximations of functions on a compact interval I, we noticed that when y is not in $L^2(R)$ the discrete least square method was significantly better than projecting $1_I y$ orthogonal to V_j (with the indicator function 1_I). This method even gives very good approximations when using relatively few basis elements. In this paper we show that the orthogonal projection from $1_I y$ to V_j with the Shannon wavelet leads approximately to a Fourier series with a Gibbs effect. Because we cut a piece of y with $1_I y$ it has consequences in the Fourier space so that even if y is in $L^2(R)$ and its Fourier transform would have a good decay behavior, $1_I y$ has than a worse the decay. But if we continue $1_I y$ so, that the continuation of y is in $L^2(R)$, we get a good approximation of y on I with the orthogonal projection.

Introduction

In the wavelet theory a scaling function ϕ is used, which belongs to a MSA (multi scale analysis). From the MSA we know, that we can construct an orthonormal basis of a closed subspace V_j, where V_j belongs to a the sequence of subspaces with the following property:

$$\ldots \subset V_{-1} \subset V_0 \subset V_1 \subset \ldots \subset L^2(R),$$

$\{\phi_{j,k}(t)\}_{k\in Z}$ is an orthonormal basis of V_j with $\phi_{j,k}(t) = 2^{j/2}\phi(2^j t - k)$.

We want to get an approximation of a function y on a compact interval I. So y must not be in $L^2(R)$ but only in $L^2(I)$.

We use the approximation function

$$y_j(t) := \sum_{k=k_{\min}}^{k_{\max}} c_k \cdot \phi_{j,k}(t) \quad .$$

The Approximation

In the following example we will see, that if we calculate the coefficients c_k by the minimization of

$$Q(c) = \sum_{i=0}^{m} \left(y_j(t_i) - y(t_i) \right)^2 \tag{1}$$

with $t_i \in I$ it will lead to much better results than if we calculate the coefficients c_k with the orthogonal projection from $1_I y$ on V_j:

$$c_k = \left\langle 1_I\, y, \phi_{j,k} \right\rangle = \int_I y(t) \cdot \phi_{j,k}(t)\, dt \tag{2}$$

Here c_k depends on j too, but for easier notation we write short c_k. This is an analogous result to [1], where we used an ODE and we minimized the residuals instead of (1). A reason for the worse approximation is, that a function with compact support like $1_I\, y$ is not very concentrated on special frequency areas in the Fourier space, that means the Fourier transform of $1_I\, y$ has not a good decay behaviour. For example $1_{[-1,1]}$ has the Fourier transform

$$Y(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{-i\omega t} dt = \sqrt{\frac{2}{\pi}} \cdot \frac{\sin(\omega)}{\omega} \quad .$$

Because of the multiplication theorem the Fourier transform of $1_{[-1,1]} \bullet^n$ (with $n \in N$) are derivatives of the Y function multiplied with i^n and have the same decay behaviour as Y. And generally, $1_I\, y$ has jumps at the points at the edges of the interval I, which leads to a poorer decay behaviour of the Fourier transform (see remarks). If we continue $1_I\, y$, so that the completion $\tilde{y}$ is quadratic integrabel on R, we get a much better approximation if we calculate the orthogonal projection form that completion $\tilde{y}$ on V_j. Here we use the function

$$h(t) = e^{-t^2}$$

and set (for $I = [a, b]$):

(3) $$\tilde{y}(t) = \begin{cases} h(t-a)\cdot y(a) & if \quad t < a \\ y(t) & if \quad a \le t \le b \\ h(t-b)\cdot y(b) & if \quad t > b \end{cases}$$

We can show, that $1_I\, y$ on V_j can be approximated with a partial sum of a Fourier series, if we use the Shannon wavelet. The reason is, that if we calculate the inverse Fourier transform of $1_I\, y$ we get a function $g = \mathcal{F}^{-1}(1_I y)$, which has a compact support. Here we swap original space with Fourier space for the direct application of the Shannon theorem. Because of the Shannon theorem we know, that if $I \subseteq [-2^r \cdot \pi, 2^r \cdot \pi]$ than $g \in V_r$ and

(4) $$g(s) = \sum_{k-\infty}^{\infty} g_k \cdot \phi_{r,k}(s) \quad \text{with } g_k = 2^{-r/2} g(k/2^r)$$

for almost all s. So the coefficients g_k can be written as function values.

If we calculate the Fourier transform of g we get

(5) $$G(t) = \frac{2^{-r/2}}{\sqrt{2\pi}} \cdot \sum_k g_k \cdot 1_{[-2^r\pi, 2^r\pi]}(t) \cdot e^{-itk/2^r} \quad .$$

This is a Fourier series of $1_I\, y$ (with respect to the interval $[-2^r \cdot \pi, 2^r \cdot \pi]$) and

(6) $$G_{n_{min}, n_{max}}(t) = \frac{2^{-r/2}}{\sqrt{2\pi}} \cdot \sum_{k=n_{min},\dots,n_{max}} g_k \cdot 1_{[-2^r\pi, 2^r\pi]}(t) \cdot e^{-itk/2^r}$$

is the approximation. Now we need a big summation area $[n_{min}, n_{max}]$ to get with y_j a good approximation of $1_I\, y$, if g has a worse decay behaviour and so a big j, because $\mathrm{supp}(Y_j) = [-2^j\cdot\pi,\ 2^j\cdot\pi]$. Thus with respect to y_j we can only considers the g_k with $|k|/2^r \in [-2^j\cdot\pi,\ 2^j\cdot\pi]$ and n_{min} is the smallest integer n with $n \geq -2^{j+r}\cdot\pi$ and n_{max} the biggest integer n with $n \leq 2^{j+r}\cdot\pi$, if $G_{n_{min},n_{max}}$ should be in V_j. Then, with growing r, the function $G_{n_{min},n_{max}}$ tends to the Fourier transform of $g\cdot 1_{[-2^j\pi, 2^j\pi]}$, i.e. to y_j. This is the reason for a worse approximation of $1_I\, y$ with an orthogonal projection on V_j with a small j.

Example:
In this example we use the Shannon wavelet. We want to approximate $y(t) = e^{-t}$ on $I = [0, 1]$. The orthogonal projection from $1_I\, y$ on V_3 leads to a worse approximation, what we can see on the graph of y_3 ($-k_{min} = k_{max} = 24$):

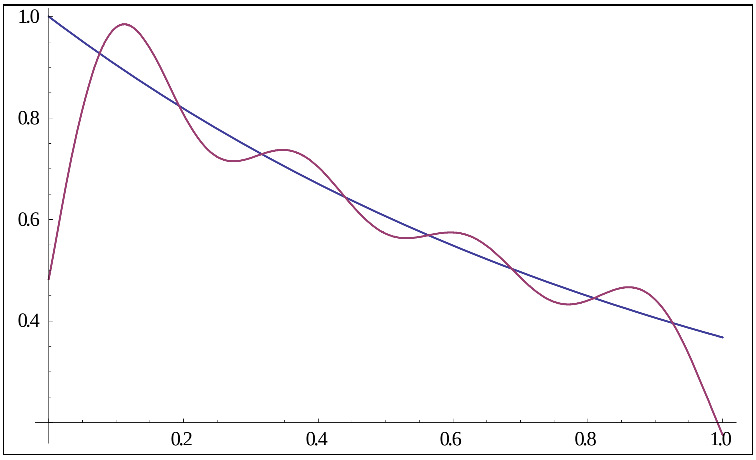

Figure 1. Graphs of y and of y_3 ($-k_{min} = k_{max} = 24$) with $c_k = \langle 1_{[0,1]}\, y, \phi_{j,k}\rangle$

$$\|y - y_3\|_{L^2([0,1])} \approx 0.0881117$$
$$\|y - y_3\|_{\infty} \approx 0.517282 \text{ (on } I\text{)}$$

In the next figure we use instead of $y(t) = e^{-t}$ the function $m(t) = e^{-|t|}$ and if we calculate the orthogonal projection from $1_{[-3,3]}\cdot m$ on V_3 we get (we set $-k_{min} = k_{max} = 24$):

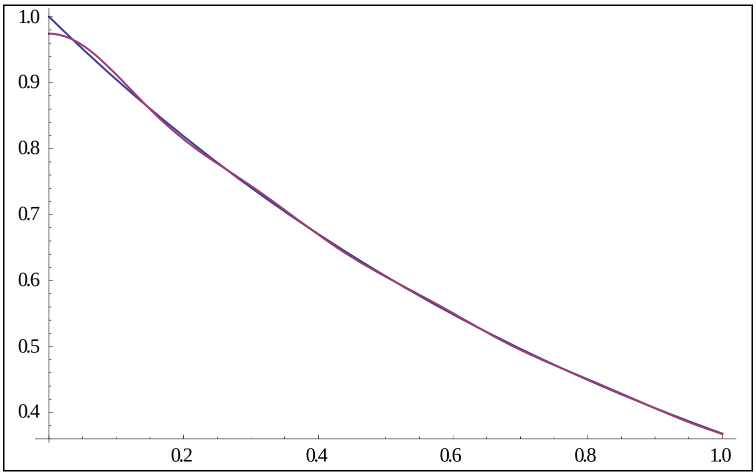

Figure 2. Graphs of y and of y_3 ($-k_{min} = k_{max} = 24$) with $c_k = \langle 1_{[-3,3]}\, m, \phi_{j,k}\rangle$

$$\|y - y_3\|_{L^2([0,1])} \approx 0.00369947$$

$$\|y - y_3\|_\infty \approx 0.0257361 \text{ (on } I\text{)}$$

If we continue $1_I y$ like in (3) with $a = 0$ and $b = 1$ then we get the following graph of $\tilde{y}$:

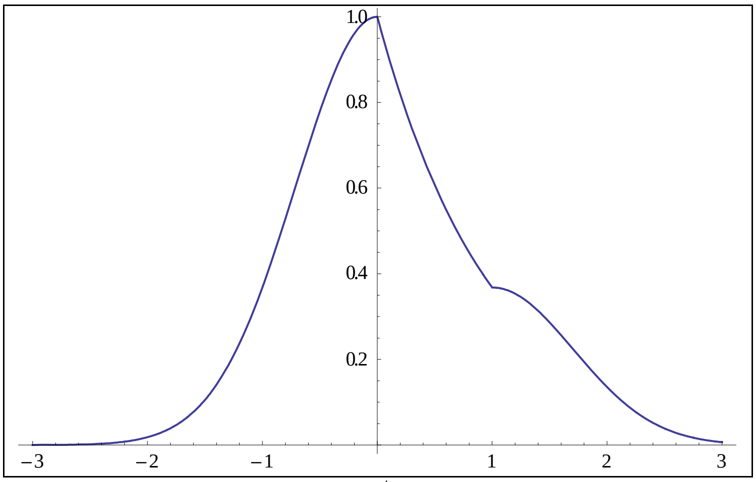

Figure 3. Graph of $\tilde{y}$ with $y(t) = e^{-t}$ and $a = 0$ and $b = 1$

Now we calculated the orthogonal y_3 projection of $1_{[-3,3]} \cdot \tilde{y}$ on V_3 (we can also project $\tilde{y}$ on V_3, but in the practical case we integrate only over a finite interval). Here we see the graph of $\tilde{y}$ and y_3 (with $-k_{min} = k_{max} = 24$):

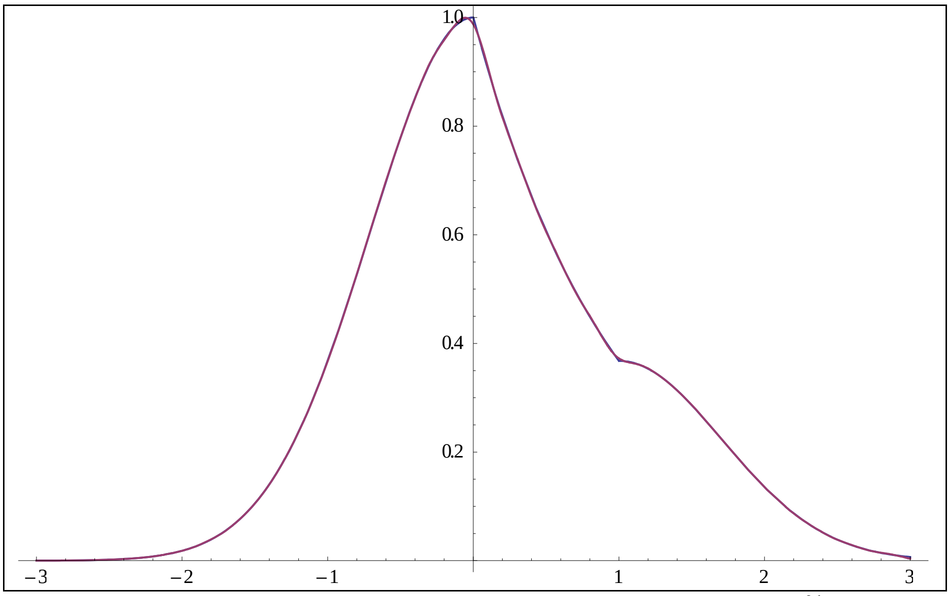

Figure 4. Graphs of $\tilde{y}$ and the orthogonal projection from $1_{[-3,3]} \cdot \tilde{y}$ on V_3

$$\|y - y_3\|_{L^2([0,1])} \approx 0.0020488$$

$$\|y - y_3\|_\infty \approx 0.0126504 \text{ (on } I\text{)}$$

With a differentiable $\tilde{y}$ we could get with a smaler j even better results.

The graph of the amplitude spectrum of $1_{[0,1]}\, y$ (the absolute values of the Fourier transform of $1_{[0,1]}\, y$):

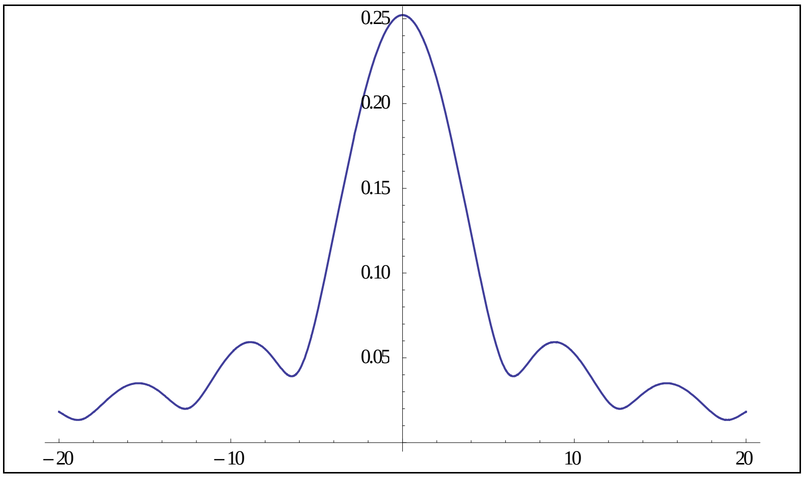

Figure 5. Graph of the amplitude spectrum of $1_{[0,1]}\, y$

And here the graph of the amplitude spectrum of $1_{[-3,3]} \cdot \tilde{y}$ (the absolute values of the Fourier transform of $1_{[-3,3]} \cdot \tilde{y}$):

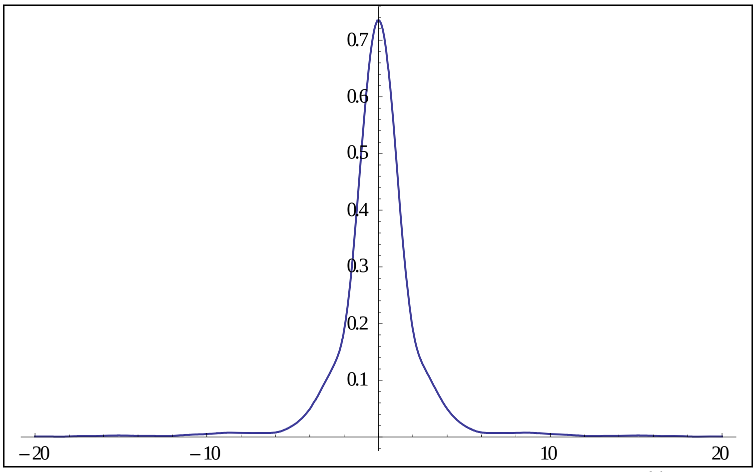

Figure 6. Graph of the amplitude spectrum of $1_{[-3,3]} \cdot \tilde{y}$

We can see that Fourier transform of $1_{[-3,3]} \cdot \tilde{y}$ has a much better decay behaviour as $1_{[0,1]}\, y$ which is the reason for a better approximation if we use the orthogonal projection on V_j with a small j as an approximation function.

Here is the graph of abs(g) (r = 0, because $[0,1] \subseteq [-2^r \cdot \pi, 2^r \cdot \pi]$, we could even set r = -1) for $1_{[0,1]} y$ ($g = \mathcal{F}^{-1}(1_I\, y)$ and the band limited function g can be written as a linear combination of bases elements of V_0, see (4)).

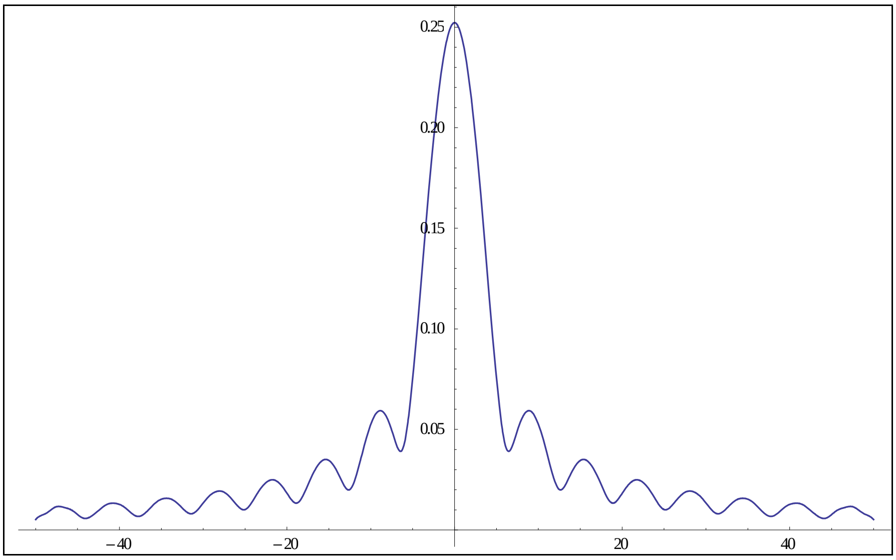

Figure 7. Graph of abs(g)

The Fourier transform of g is a Fourier series of $1_{[0,1]}\,y$. Here is the graph of $G_{n_{min},n_{max}}$ (see (6)) and $1_{[0,1]}\,y$ for $-n_{min} = n_{max} = 50$:

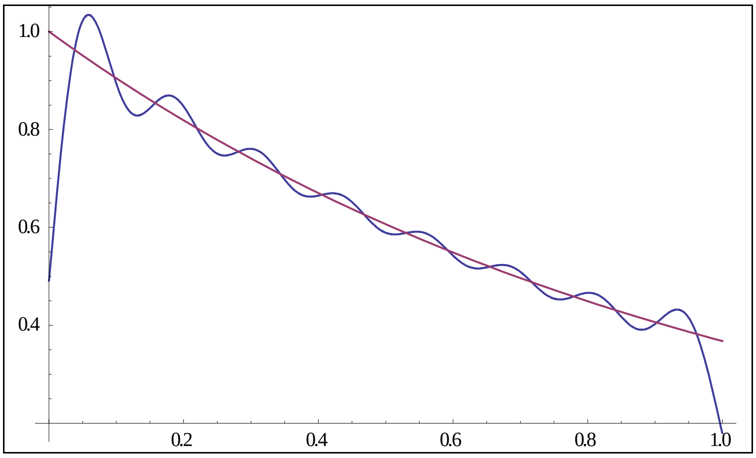

Figure 8. Graphs $1_{[0,1]}\,y$ and $G_{-50,50}$

$G_{-50,50}$ is a good approximation of the orthogonal projection from $1_{[0,1]}\,y$ on V_4 (because $-n_{min} = n_{max}$ is near $2^4\cdot\pi = 2^{r+j}\cdot\pi$), although r is not very big. Here is the graph of $G_{-100,100}$ and $1_{[0,1]}\,y$:

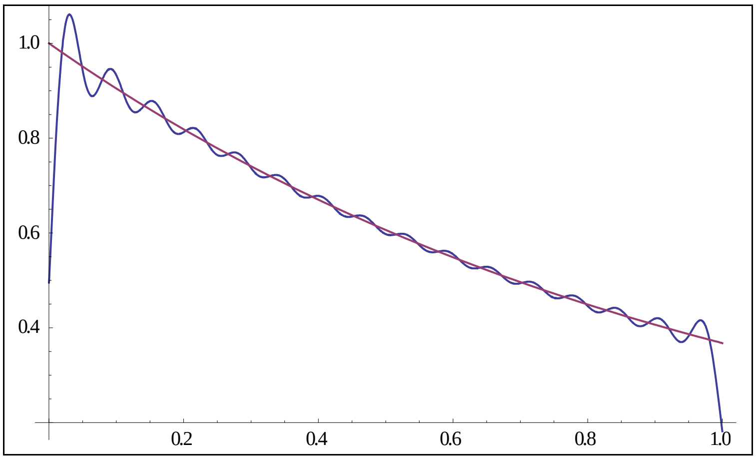

Figure 9. Graphs $1_{[0,1]}\,y$ and $G_{-100,100}$

Above we see the Gibbs effect and we see that we need a big summation area.

Here is the scheme of the transformation and projection:

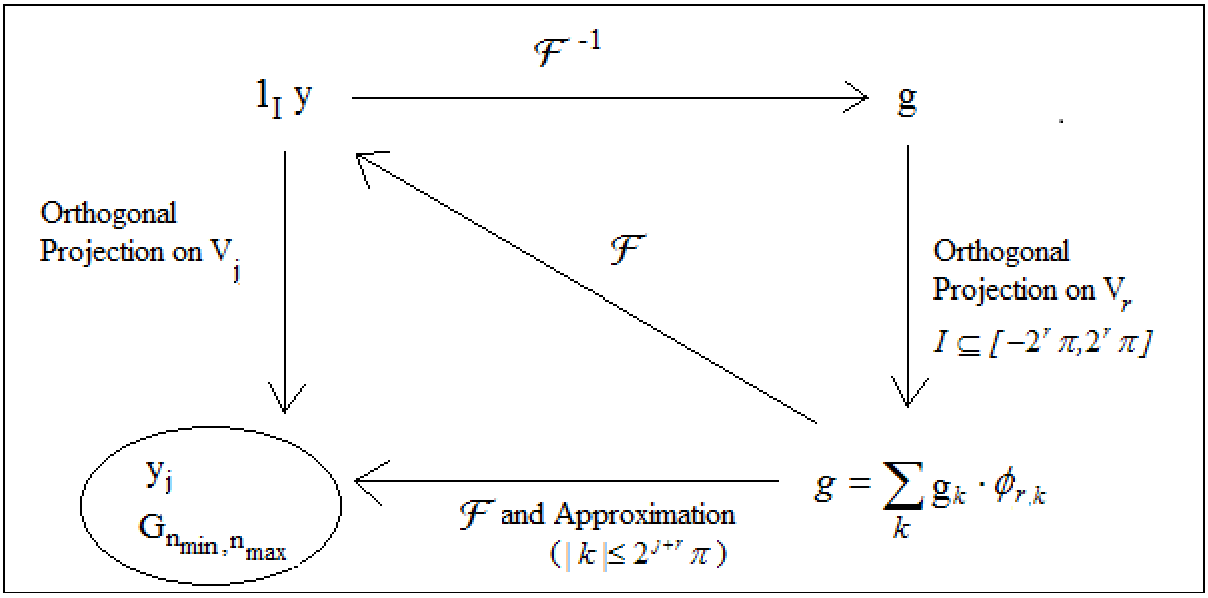

Figure 10. Scheme of the transformation and projection

At least we calculate a discrete approximation of y on $I = [0, 1]$. Here we set $-k_{min} = k_{max} = 10$ and we minimize Q (see (1)) with $j = 0$. We use $m = 20$ and $t_i = i/20$. That leads to a very good approximation.

Here are the graphs of y and y_0:

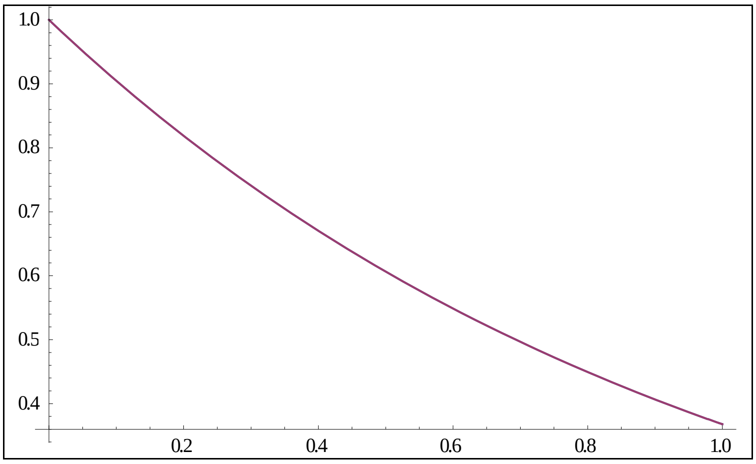

Figure 11. Graphs of y and y_0

Here is the graph of the error $y_0 - y$:

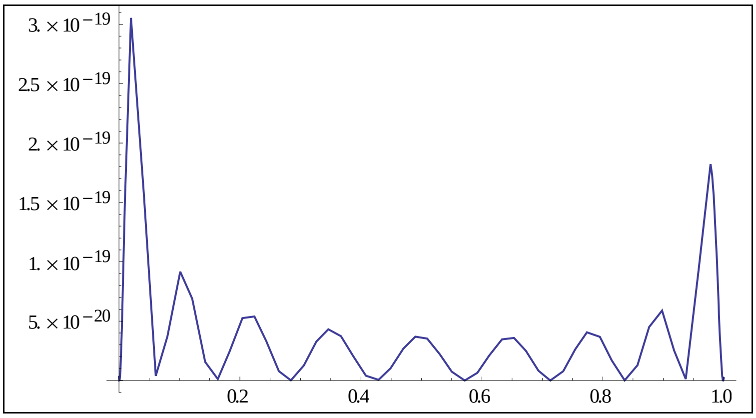

Figure 12. Graph of the error y_0 - y

Remarks:

1) Some authors suggest a regression, where the coefficients c_k are calculated over the scalar product $c_k = \langle \tilde{y}, \phi_{j,k} \rangle$ with a help function $\tilde{y}$ through the points (t_i, y_i) (with $i = 1, 2, ..., n$ and $t_i \in [a, b]$) and with supp $\tilde{y} = [a, b]$. Because generally the help function is not continuous we have the same problem of bad decay behaviour of the coefficients c_k and we also need a big j.

2) If y has a jump at the point t = ξ then we get for the Fourier transform of y (if $y' \in \mathcal{L}^1(R)$):

$$Y(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} y(t) \cdot e^{-i\cdot\omega\cdot t} dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\xi^-} y(t) \cdot e^{-i\cdot\omega\cdot t} dt + \frac{1}{\sqrt{2\pi}} \int_{\xi^+}^{\infty} y(t) \cdot e^{-i\cdot\omega\cdot t} dt$$

$$= \frac{1}{\sqrt{2\pi}} \left(\left[y(t) \cdot \frac{1}{-i\cdot\omega} e^{-i\cdot\omega\cdot t} \right]_{-\infty}^{\xi^-} + \int_{-\infty}^{\xi^-} y'(t) \cdot \frac{1}{i\cdot\omega} e^{-i\cdot\omega\cdot t} dt + \left[y(t) \cdot \frac{1}{-i\cdot\omega} e^{-i\cdot\omega\cdot t} \right]_{\xi^+}^{\infty} + \int_{\xi^+}^{\infty} y'(t) \cdot \frac{1}{i\cdot\omega} e^{-i\cdot\omega\cdot t} dt \right)$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{y(\xi^+) - y(\xi^-)}{i\cdot\omega} e^{-i\cdot\omega\cdot\xi} + \int_{R\setminus\{\xi\}} y'(t) \cdot \frac{1}{i\cdot\omega} e^{-i\cdot\omega\cdot t} dt \right)$$

$$= \frac{1}{\sqrt{2\pi}} \left(O\left(\frac{1}{\omega}\right) + \int_{R\setminus\{\xi\}} y'(t) \cdot \frac{1}{i\cdot\omega} e^{-i\cdot\omega\cdot t} dt \right)$$

References

[1] M. Schuchmann, M. Rasguljajew. *An Approximation on a Compact Interval Calculated with a Wavelet Collocation Method can Lead to Much Better Results than other Methods.* Journal of Approximation Theory and Applied Mathematics (2013, Vol. 1)

[2] M. Schuchmann. *Approximation and Collocation with Wavelets. Approximations and Numerical Solving of ODEs, PDEs and IEs.* Osnabrück: DAV, (2012).