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ADAPTED LINEAR APPROXIMATION FOR LOGARITHMIC KERNEL INTEGRALS

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ABSTRACT. In this work we present an approximation for singular integrals with logarithmic kernel on a smooth oriented contour, for this latter we use a small modification of the linear spline functions in order to eliminate the weak singularity. Noting that this approximation is destined to solve numerically the integral equations with weakly singular kernel on a smooth oriented contour.

Introduction

As we know Fredholm integral equations of the second kind appear in many applications among those transport theory, potential theory, elasticity and crack mechanics. All those areas of mathematical physics contain a problems lead to the equation of the form

$$\varphi(t_0) + \frac{1}{\pi i}\int_\Gamma k(t,t_0)\varphi(t)dt = f(t_0), \tag{1}$$

for a given function $f(t_0)$ and a given singular kernel $k(t,t_0)$ of the type

$$k(t,t_0) = h(t,t_0)\ln(t-t_0),$$

where $h(t,t_0)$ is non singular function and under Γ we designate an oriented smooth contour, the points t and t_0 are on Γ.

Our problem describe a new approximation of singular integral with logarithmic kernel

$$F(t_0) = \frac{1}{\pi i}\int_\Gamma \varphi(t)\ln(t-t_0)dt, \quad t,t_0 \in \Gamma. \tag{2}$$

Noting that, this integral can be derived when we use the single layer potential for a boundary element method of Laplace equation with the Dirichlet boundary data. for the existence of the principal value of this integral for a given density $\varphi(t)$, we will need more than mere

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1

continuity. In other words, the density $\varphi(t)$ has to satisfy the Holder condition $H(\mu)[1-2-3]$.

The function $\varphi(t)$ will be said to satisfy a Holder condition on Γ, if for any two points t_1 and t_2 of Γ

$$\mid \varphi(t_2) - \varphi(t_1) \mid \leq A \mid t_2 - t_1 \mid^{\mu}, \quad 0 < \mu \leq 1,$$

where A is a positive constant, called the Holder constant and μ the Holder index.

The Quadrature

We denote by t the parametric complex function $t(s)$ of the curve Γ defined by

$$t(s) \ = x(s) + iy(s), \quad a \leq s \leq b,$$

where $x(s)$ and $y(s)$ are continuous functions on the finite interval of definition $[a,b]$ and have continuous first derivatives $x'(s)$ and $y'(s)$ never simultaneously null. Let N be an arbitrary natural number, generally we take it large enough and divide the interval $[a,b]$ into N equal subintervals $I_1, I_2, ..., I_N$ by the points

$$s_\sigma = a + \sigma \frac{l}{N} \ , \ l = b - a \ , \ \sigma = 0, 1, 2, ..., N.$$

Assuming that, for the indices $\sigma, \nu = 0, 1, 2, ..., N-1$, the points t and t_0 belong respectively to the arcs $t_\sigma \overset{\frown}{t_{\sigma+1}}$ and $t_\nu \overset{\frown}{t_{\nu+1}}$ where $t_\alpha \overset{\frown}{t_{\alpha+1}}$ designates the smallest arc with ends t_α and $t_{\alpha+1}$ $[4-5-6]$.

For an arbitrary number $\sigma = 0, 1, 2, ..., N-1$, we define the linear spline interpolation polynomial $S_1(\varphi; t, \sigma)$ dependent on φ, t and σ which represents the linear approximation of the function density $\varphi(t)$ on the subinterval $[t_\sigma, t_{\sigma+1}]$ of the curve Γ, given by the following equation

For $t_\sigma \leq t \leq t_{\sigma+1}$,

$$S_1(\varphi; t, \sigma) = \frac{(t_{\sigma+1} - t)}{(t_{\sigma+1} - t_\sigma)} \varphi(t_\sigma) + \frac{(t - t_\sigma)}{(t_{\sigma+1} - t_\sigma)} \varphi(t_{\sigma+1}). \tag{3}$$

This spline function exists and is unique also called a linear interpolating polynomial.

We define for an arbitrary numbers σ and ν such that $0 \leq \sigma, \nu \leq N-1$, the function $\beta_{\sigma\nu}(\varphi; t, t_0, \sigma, \nu)$ dependents of φ, t and t_0

$$\beta_{\sigma\nu}(\varphi; t, t_0, \sigma, \nu) = \begin{cases} U(\varphi; t, t_0, \sigma) - V(\varphi; t, t_0, \sigma, \nu) & \text{if} \quad t \neq t_0 \\ 0 & \text{if} \quad t = t_0 \end{cases} \tag{4}$$

Journal of Approximation Theory and Applied Mathematics, 2014 Vol. 3

The function $U(\varphi; t, t_0, \sigma)$ represents a modified linear interpolation of the function density $\varphi(t)$ on the subinterval $[t_\sigma, t_{\sigma+1}]$ of the curve Γ. Indeed, for $t_\sigma \leq t \leq t_{\sigma+1}$ and $t - t_0 \neq 1$, we put

$$\begin{aligned} U(\varphi; t, t_0, \sigma) &= \frac{(t_{\sigma+1} - t)}{(t_{\sigma+1} - t_\sigma)} \varphi(t_{\sigma k}) \frac{\ln(t_\sigma - t_0)}{\ln(t - t_0)} \\ &+ \frac{(t - t_\sigma)}{(t_{\sigma+1} - t_\sigma)} \varphi(t_{\sigma+1}) \frac{\ln(t_{\sigma+1} - t_0)}{\ln(t - t_0)}, \end{aligned}$$

and the function $V(\varphi; t, t_0, \sigma, \nu)$ is given by

$$\begin{aligned} V(\varphi; t, t_0, \sigma, \nu) &= S_1(\varphi; t_0, \nu) \frac{\ln(t_\sigma - t_0)}{\ln(t - t_0)} \frac{(t_{\sigma+1} - t)}{(t_{\sigma+1} - t_\sigma)} \\ &+ S_1(\varphi; t_0, \nu) \frac{\ln(t_{\sigma+1} - t_0)}{\ln(t - t_0)} \frac{(t - t_\sigma)}{(t_{\sigma+1} - t_\sigma)}, \end{aligned}$$

where the function φ represents a given function on the curve Γ of the class $H\ (\mu)$.

Denoting by $\psi_{\sigma\nu}(\varphi; t, t_0, \sigma, \nu)$ the quadratic approximation of the density $\varphi(t)$ at the point $t \in [t_\sigma, t_{\sigma+1}]$, $t_0 \in [t_\nu, t_{\nu+1}]$ and $0 \leq \sigma, \nu \leq N - 1$ by

$$\psi_{\sigma\nu}(\varphi; t, t_0, \sigma, \nu) = \varphi(t_0) + \beta_{\sigma\nu}(\varphi; t, t_0, \sigma, \nu), \tag{5}$$

and replacing the density $\varphi(t)$ by expansion (5) in the weakly singular integral (2)

$$F(t_0) = \frac{1}{\pi i} \int\limits_\Gamma \varphi(t) \ln(t - t_0) dt,$$

and obtain the following approximation noting by $S(\varphi; t)$ given as

$$\begin{aligned} S(\varphi; t_0) &= \frac{1}{\pi i} \int_\Gamma \psi_{\sigma\nu}(\varphi; t, t_0, \sigma, \nu) \ln(t - t_0) dt \\ &= \frac{1}{\pi i} \int_\Gamma \beta_{\sigma\nu}(\varphi; t, t_0, \sigma, \nu) \ln(t - t_0) dt. \end{aligned} \tag{6}$$

Main results

Theorem

Let Γ be an oriented smooth contour and let φ be a density function defined on Γ and satisfying *the Holder condition $H(\mu)$ then, the*

Journal of Approximation Theory and Applied Mathematics, 2014 Vol. 3

following estimation

$$| F(t_0) - S(\varphi; t_0) | \leq \frac{C \ln N}{N^{\mu}} \quad N > 1$$

holds, where the constant C depends only of the contour Γ and the Holder's constant.

Proof

For any points $t \in [t_\sigma, t_{\sigma+1}]$ and $t_0 \in [t_\nu, t_{\nu+1}]$ we have

$$\begin{aligned}
\varphi(t) - \psi_{\sigma\nu}(\varphi; t, t_0, \sigma, \nu) &= \varphi(t) - \varphi(t_0) \\
&-\{\frac{t_{\sigma+1} - t}{t_{\sigma+1} - t_\sigma}\varphi(t_\sigma)\frac{\ln(t_\sigma - t_0)}{\ln(t - t_0)} \\
&+\frac{t - t_\sigma}{t_{\sigma+1} - t_\sigma}\varphi(t_{\sigma+1})\frac{\ln(t_{\sigma+1} - t_0)}{\ln(t - t_0)} \\
&-\frac{S_1(\varphi; t_0, \nu)\ln(t_\sigma - t_0)(t_{\sigma+1} - t)}{\ln(t - t_0)(t_{\sigma+1} - t_\sigma)} \\
&-\frac{S_1(\varphi; t_0, \nu)\ln(t_{\sigma+1} - t_0)(t - t_\sigma)}{\ln(t - t_0)(t_{\sigma+1} - t_\sigma)}\}.
\end{aligned} \tag{7}$$

Taking into account the expression (7) we get

$$\begin{aligned}
\int_\Gamma \ln(t - t_0)(\varphi(t) - \psi_{\sigma\nu}(\varphi; t, t_0, \sigma, \nu))dt &= \sum_{\sigma=0}^{N-1} \int_{t_\sigma t_{\sigma+1}} (\varphi(t) - \varphi(t_0))\ln(t - t_0) \\
&-\beta_{\sigma\nu}(\varphi; t, t_0, \sigma, \nu)\ln(t - t_0)dt,
\end{aligned} \tag{8}$$

hence

$$\begin{aligned}
F(t_0) - S(\varphi; t_0) &= \frac{1}{\pi i}\sum_{\sigma=0}^{N-1} \int_{t_\sigma t_{\sigma+1}} (\varphi(t) - \varphi(t_0))\ln(t - t_0) \\
&-\{\frac{t_{\sigma+1} - t}{t_{\sigma+1} - t_\sigma}\varphi(t_\sigma)\frac{\ln(t_\sigma - t_0)}{\ln(t - t_0)} \\
&+\frac{t - t_\sigma}{t_{\sigma+1} - t_\sigma}\varphi(t_{\sigma+1})\frac{\ln(t_{\sigma+1} - t_0)}{\ln(t - t_0)} \\
&-\frac{S_1(\varphi; t_0, \nu)\ln(t_\sigma - t_0)(t_{\sigma+1} - t)}{\ln(t - t_0)(t_{\sigma+1} - t_\sigma)} \\
&-\frac{S_1(\varphi; t_0, \nu)\ln(t_{\sigma+1} - t_0)(t - t_\sigma)}{\ln(t - t_0)(t_{\sigma+1} - t_\sigma)}\}\ln(t - t_0)dt.
\end{aligned}$$

Seeing that, the equalities $t_\sigma - t_0 = 0$ and $t_{\sigma+1} - t_0$ are possible only when $\sigma = \nu - 1, \nu + 1$and ν. For the two first cases the integral (8) exists when t_σ tends to t_0 or $t_{\sigma+1}$ tends to t_0; the other case, if $\sigma = \nu$ we can easily seeing that, the function $\beta_{\sigma\sigma}(\varphi; t, t_0, \sigma, \sigma)$ contains $(t_\sigma - t_0)$ and $(t_{\sigma+1} - t_0)$ as factor, for the points $t,\ t_0 \in [t_\sigma, t_{\sigma+1}]$ we

Journal of Approximation Theory and Applied Mathematics, 2014 Vol. 3

write

$$\beta_{\sigma\sigma}(\varphi;t,t_0,\sigma,\sigma)=U(\varphi;t,t_0,\sigma)-V(\varphi;t,t_0,\sigma,\sigma),$$

hence

(9)

$$\begin{aligned}\beta_{\sigma\sigma}(\varphi;t,t_0,\sigma,\sigma)=&\frac{(t_{\sigma+1}-t)\ln(t_\sigma-t_0)}{(t_{\sigma+1}-t_\sigma)\ln(t-t_0)}(\varphi(t_\sigma)-S_1(\varphi;t_0,\sigma))\\&+\frac{(t-t_\sigma)\ln(t_{\sigma+1}-t_0)}{(t_{\sigma+1}-t_\sigma)\ln(t-t_0)}(\varphi(t_{\sigma+1})-S_1(\varphi;t_0,\sigma)).\end{aligned}$$

In other words, we write

$$\beta_{\sigma\sigma}(\varphi;t,t_0,\sigma,\sigma)=\frac{1}{\ln(t-t_0)}Q(\varphi;t,t_0,\sigma,\sigma),$$

where the expression $Q(\varphi;t,t_0,\sigma,\sigma)$ is given by

$$\begin{aligned}Q(\varphi;t,t_0,\sigma,\sigma)=&\frac{(t_{\sigma+1}-t)\ln(t_\sigma-t_0)}{(t_{\sigma+1}-t_\sigma)}(\varphi(t_\sigma)-S_1(\varphi;t_0,\sigma))\\&+\frac{(t-t_\sigma)\ln(t_{\sigma+1}-t_0)}{(t_{\sigma+1}-t_\sigma)}(\varphi(t_{\sigma+1})-S_1(\varphi;t_0,\sigma)),\end{aligned}$$

with

$$\varphi(t_\sigma)-S_1(\varphi;t_0,\sigma)=\frac{(t_0-t_\sigma)}{(t_{\sigma+1}-t_\sigma)}(\varphi(t_\sigma)-\varphi(t_{\sigma+1})),$$

and

$$\varphi(t_{\sigma+1})-S_1(\varphi;t_0,\sigma).=\frac{(t_{\sigma+1}-t_0)}{(t_{\sigma+1}-t_\sigma)}(\varphi(t_{\sigma+1})-\varphi(t_\sigma))$$

Passing now to the estimation of the expression (8).

For $t\in[t_\sigma,t_{\sigma+1}]$ and $t_0\in[t_\nu,t_{\nu+1}]$ with the conditions $\sigma\neq\nu-1,\nu+1$ and ν we have

$$\left|\begin{array}{l}\frac{1}{\pi i}\sum_{\sigma=0}^{N-1}\int_{t_\sigma t_{\sigma+1}}(\varphi(t)-\varphi(t_0))\ln(t-t_0)\\-\{\frac{t_{\sigma+1}-t}{t_{\sigma+1}-t_\sigma}\varphi(t_\sigma)\frac{\ln(t_\sigma-t_0)}{\ln(t-t_0)}\\+\frac{t-t_\sigma}{t_{\sigma+1}-t_\sigma}\varphi(t_{\sigma+1})\frac{\ln(t_{\sigma+1}-t_0)}{\ln(t-t_0)}\\-\frac{S_1(\varphi;t_0,\nu)\ln(t_\sigma-t_0)(t_{\sigma+1}-t)}{\ln(t-t_0)(t_{\sigma+1}-t_\sigma)}\\-\frac{S_1(\varphi;t_0,\nu)\ln(t_{\sigma+1}-t_0)(t-t_\sigma)}{\ln(t-t_0)(t_{\sigma+1}-t_\sigma)}\}\ln(t-t_0)dt\end{array}\right|=O(\frac{\ln N}{N^\mu}).$$

Indeed, it is clear that

Journal of Approximation Theory and Applied Mathematics, 2014 Vol. 3

$$\max_{t_0 \in \widehat{t_\nu t_{\nu+1}}} \mid \frac{1}{\pi i} \sum_{\sigma=0}^{N-1} \int_{t_\sigma}^{t_{\sigma+1}} (\varphi(t) - \varphi(t_0)) \ln(t - t_0) dt \mid = O(\frac{\ln N}{N^\mu})$$

and also we estimate the expression

$$\left| \begin{array}{l} \frac{1}{\pi i} \sum_{\sigma=0}^{N-1} \int_{t_\sigma t_{\sigma+1}} -\{ \frac{t_{\sigma+1} - t}{t_{\sigma+1} - t_\sigma} \varphi(t_\sigma) \frac{\ln(t_\sigma - t_0)}{\ln(t - t_0)} \\ + \frac{t - t_\sigma}{t_{\sigma+1} - t_\sigma} \varphi(t_{\sigma+1}) \frac{\ln(t_{\sigma+1} - t_0)}{\ln(t - t_0)} \\ - \frac{S_1(\varphi; t_0, \nu) \ln(t_\sigma - t_0)(t_{\sigma+1} - t)}{\ln(t - t_0)(t_{\sigma+1} - t_\sigma)} \\ - \frac{S_1(\varphi; t_0, \nu) \ln(t_{\sigma+1} - t_0)(t - t_\sigma)}{\ln(t - t_0)(t_{\sigma+1} - t_\sigma)} \} \ln(t - t_0) dt, \end{array} \right|$$

$$\simeq \left| \frac{1}{\pi i} \sum_{\sigma=0}^{N-1} \int_{t_\sigma t_{\sigma+1}} \frac{\varphi(t_\nu) - \varphi(t_\sigma)}{t_\nu - t_\sigma} + \frac{\varphi(t_{\nu+1}) - \varphi(t_{\sigma+1})}{t_{\nu+1} - t_{\sigma+1}} dt \right| = O(\frac{\ln N}{N^\mu}).$$

Naturally, the estimation given above is obtained by using the density φ, as an element of the Holder space $H(\mu)$[2], and the following natural estimation

$$\left| \frac{\ln(t_\sigma - t_0)}{\ln(t - t_0)} \right| \simeq \left| \frac{\ln(t_{\sigma+1} - t_0)}{\ln(t - t_0)} \right| = O(1).$$

Further, for the cases where $\sigma = \nu - 1, \nu + 1$ and ν, using the relation (9) and the smoothness of Γ, we obtain

$$\begin{aligned} \mid \int_{t_\nu t_{\nu+1}} (\varphi(t) - \varphi(t_0)) \ln(t - t_0) dt \mid &\leq A \int_{s_\nu}^{s_{\nu+1}} \mid s - s_0 \mid^\mu |\ln(s - s_0)| \, ds \\ &= O(\frac{\ln N}{N^{(\mu+1)}}) \end{aligned}$$

Numerical experiments

Using our approximation, we apply the algorithms to singular integrals and we present results concerning the accuracy of the calculations, in this numerical experiments each table I represents the exact weakly singular integral and $\widetilde{I}$ corresponds to the approximate calculation produced by our approximation at points values interpolation.

Example 1

Consider the weakly singular integral,

$$I = F(t_0) = \int_\Gamma \varphi(t) \ln(t - t_0) dt,$$

where the curve Γ designate the unit circle and the function density φ is given by the following expression

$$\varphi(t) = -\frac{1}{t^2}.$$

N	$\parallel I - \widetilde{I} \parallel_1$	$\parallel I - \widetilde{I} \parallel_2$	$\parallel I - \widetilde{I} \parallel_\infty$
20	5.7174414E-02	3.5285469E-02	2.9367447E-02
30	5.5649281E-03	2.9125938E-03	1.8216968E-03
40	1.4109910E-03	7.3831965E-04	5.1295757E-04

Example 2

Consider the weakly singular integral,

$$I = F(t_0) = \int_\Gamma \varphi(t) \ln(t - t_0) dt,$$

where the curve Γ designate the unit circle and the function density φ is given by the following expression

$$\varphi(t) = \frac{2}{t^3}.$$

N	$\parallel I - \widetilde{I} \parallel_1$	$\parallel I - \widetilde{I} \parallel_2$	$\parallel I - \widetilde{I} \parallel_\infty$
20	1.5601690E-01	9.2777811E-02	6.5414310E-02
30	1.5781343E-02	9.7699165E-03	6.8665147E-03
40	2.1157265E-03	1.1530236E-03	7.7348948E-04

Note Many examples confirm the efficiency of this approximation.

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