Journal of Approximation Theory and Applied Mathematics

Parameter Identification with a Wavelet Collocation Method in a Partial Differential Equation

M. Schuchmann and M. Rasguljajew from the Darmstadt University of Applied Sciences

Abstract

In this article we describe a parameter identification method for an PDE. We use a wavelet collocation method and we show in a simulation that the error of the parameter estimation and of the approximation correlates with a sum of squares of the residuals. So we can assess the approximation function and the estimated parameters. This method can be applied analogous for PDEs of higher order. In the example we use the Shannon wavelet.

Introduction of the Collocation Method

As an example we want to solve numerically a PDE (with boundary conditions) of first order

$$F(u(x,y),u_x(x,y),u_y(x,y),x,y)=0,$$

$$u(x,0)=h(x)$$

on the area $D=[a_1,b_1]\times[a_2,b_2]$. Generally we use a scaling function $\phi\in C^r$ (if the order of the PDE is less or equal r). With that scaling function we can construct a two dimensional scaling function with

$$\phi(x,y)=\phi(x)\cdot\phi(y)$$

and we get the basis elements of V_j with

$$\phi_{j,k_1,k_2}(x,y)=2^j\phi(2^j x-k_1,2^j y-k_2) \text{ and } k_1, k_2\in\mathrm{Z}.$$

With those basis elements we construct an approximation function (for easier notation we don't use the index j in g):

$$g(x,y)=\sum_{k_1=n_u}^{n_o}\sum_{k_2=m_u}^{m_o}\phi_{j,k_1,k_2}(x,y)\cdot c_{k_1,k_2}$$

Now we have $n_k=(n_o-n_u+1)\cdot(m_o-m_u+1)$ unknown coefficients c_{k_1,k_2}.

To get an approximation of the solution u we can solve the following equation :

(1a) $$F(g(x_i,y_l),g_x(x_i,y_l),g_y(x_i,y_l),x_i,y_l)=0,\ i=0,\ldots,n_1,\ l=0,\ldots,n_2$$

(1b) $$g(z_e,0)=h(z_e),\ e=0,\ldots,n_3$$

With $(x_i, y_l) \in D$, $z_e \in [a_1, b_1]$, $n_k = (n_1 + 1) \cdot (n_2 + 1) + n_3 + 1$ and the collocation points $x_i \neq x_e, y_i \neq y_e, z_i \neq z_e$ for $i \neq e$.

Instead of the equations above we solve a minimum problem:

$$min\ Q(c)$$

with

$$Q(c) = \sum_i \sum_l F(g(x_i, y_l), g_x(x_i, y_l), g_y(x_i, y_l), x_i, y_l)^2 + \sum_e (g(z_e, 0) - h(z_e))^2$$

This has the advantage that we can use more collocation points.

By using the same collocation points like in equations (1) the minimization problem is equivalent. If we use more collocation points in the minimum problem the equations (1) are only fulfilled approximately. But with a good approximation function *g* the minimum of *Q* is very small. If *F* is linear then we have a quadratic minimization problem.

We use the collocation points:

$$x_i = a_1 + i\frac{b_1 - a_1}{n_1}, \quad i = 0, \ldots, n_1;$$
$$y_l = a_2 + l\frac{b_2 - a_2}{n_2}, \quad l = 0, \ldots, n_2;$$
$$z_e = a_1 + e\frac{b_1 - a_1}{n_3}, \quad e = 0, \ldots, n_3.$$

Remark:
A possible choice of the limits of the summation n_o, n_u, m_o and m_u is such that

(2) $$\phi_{j,k_1,k_2}(x, y) = 0$$

for $(x, y) \in D$ and $k_1 > n_o, k_1 < n_u, k_2 > m_o, k_2 < m_u$. If ϕ does not have a compact support or put generally we can replace (2) with

$$\left|\phi_{j,k_1,k_2}(x, y)\right| \leq \varepsilon$$

with a suitable ε > 0. So the limmits of summation depend on the approximation area D. In the example we will see that although the scaling function of the Shannon wavelet has not a compact support we need not many bases elements for a good approximation.

For the assessment of an approximation we compare $Q_{min} = min\ Q(c)$ with

$$Q_a = \sum_i \sum_l F(g(\tilde{x}_i, \tilde{y}_l), g_x(\tilde{x}_i, \tilde{y}_l), g_y(\tilde{x}_i, \tilde{y}_l), \tilde{x}_i, \tilde{y}_l)^2 + \sum_e (g(\tilde{z}_e, 0) - h(\tilde{z}_e))^2 ,$$

$$\tilde{x}_i = a_1 + i\frac{b_1 - a_1}{a \cdot n_1}, \quad i = 0, \ldots, n_1;$$
$$\tilde{y}_l = a_2 + l\frac{b_2 - a_2}{a \cdot n_2}, \quad l = 0, \ldots, n_2;$$
$$\tilde{z}_e = a_1 + e\frac{b_1 - a_1}{a \cdot n_3}, \quad e = 0, \ldots, n_3 .$$

with an integer $a > 0$. Here g is the approximation function with c calculated by the minimization problem. In many simulations we got with $a = 2$ very good results like in example 1. The advantage of this method is that for Q_a we don't have to calculate a second estimation because we get a whole approximation function g, not only points. If Q_a is too big and Q_{min} is very small we need more collocation points.

Analogous we can solve an PDE of higher order numerically with the described method, for example

$$F(u, u_x, u_y, u_{yy}, x, y) = 0,$$

$$u(x,0) = h_1(x) \text{ and } u_y(x,0) = h_2(x) \quad ,$$

if we minimize

$$Q(c) = \sum_i \sum_l F(g(x_i, y_l), g_x(x_i, y_l), g_y(x_i, y_l), g_{yy}(x_i, y_l), x_i, y_l)^2 + \sum_e (g_y(z_e, 0) - x)^2 + \sum_e (g(z_e, 0) - 0)^2$$

The same method we now apply to a parameter identification.

Parameter Estimation and Assessment of the Approximation

We use as an example the following parameter identification problem with two parameters p_1 and p_2:

$$F(u, u_x, u_y, u_{yy}, x, y) = u_{yy} + p_1 u_y + p_2 u_x = 0,$$

$$u(x,0) = 0 \text{ and } u_y(x,0) = x \quad .$$

Additionally we need measurements $\tilde{m}_{i,l}$ from u at the points $(\tilde{x}_i, \tilde{y}_l)$.

We simulate the measurements and set $p_1 = 1$ and $p_2 = -1$, so that we get the exact solution of the PDE with boundary conditions:

$$u(x, y) = (x-2)(1-Exp(-y)) + y \cdot (1+Exp(-y)) \ .$$

We want to approximate the solution on the area $[0, 2]^2$.

Now Q depends on two vectors c and p we minimize $Q(c, p)$:

$$\min_{c,p} Q(c, p)$$

with

$$Q(c,p) = \sum_i \sum_l F(g(x_i, y_l), g_x(x_i, y_l), g_y(x_i, y_l), g_{yy}(x_i, y_l), x_i, y_l)^2 + \sum_e (g_y(z_e, 0) - x)^2 + \sum_e (g(z_e, 0) - 0)^2 + \sum_i \sum_l (\tilde{m}_{i,l} - g(\hat{x}_i, \hat{y}_l))^2$$

with $n_1 = n_2 = n_3 = 20$.

Using this method we even could identify parameters in the boundary conditions.

Now we calculate parameter estimations with $j = 0, 1, 2$ and $n_0 = m_0 = 2k_{max}$,, $n_u = m_u = - k_{max}$ and $k_{max} = 3, 4, 5, 6$. As measurement points we choose $\hat{x}_i = i \cdot 1/8, \ \ i = 0,\ldots,16$, $\hat{y}_i = i \cdot 1/8, \ \ i = 0,\ldots,16$. We use the scaling function from the Shannon wavelet, so $\phi \in C^\infty$ (see [4]).

The estimated parameter we call $\hat{p}$. Here is the table of the results:

j	k_{max}	Q_{min}	Q_2	*sse*	$\|\|p - \hat{p}\|\|$
0	3	0.0000107237	0.000041234	7.70293×10^{-9}	0.0000561938
0	4	6.21836×10^{-11}	3.89477×10^{-10}	3.25975×10^{-14}	1.16275×10^{-7}
0	5	3.52194×10^{-17}	1.54002×10^{-15}	2.63439×10^{-19}	3.19512×10^{-10}
0	6	5.16334×10^{-21}	1.15253×10^{-19}	1.50842×10^{-22}	3.16654×10^{-12}
1	3	0.0730943	0.259421	0.00405652	0.00526086
1	4	5.40941×10^{-7}	3.05754×10^{-6}	4.44273×10^{-10}	0.0000121292
1	5	3.61066×10^{-12}	1.13727×10^{-10}	2.11763×10^{-14}	1.20021×10^{-7}
1	6	5.62028×10^{-18}	1.39483×10^{-14}	2.46281×10^{-18}	1.50996×10^{-9}
2	3	259.483	275.294	899.772	5.63328
2	4	144.504	235.327	325.062	3.60919
2	5	0.217185	2.11309	0.014662	0.0157611
2	6	4.21083×10^{-8}	0.0000177721	4.62554×10^{-9}	0.0000589473

Q_{min} was calculated numerically by using the Mathematica function FindMinimum. *sse* is the error sum of squares calculated with:

$$sse = \sum_{i=0}^{32} \sum_{j=0}^{32} (g(i/16, j/16) - g(i/16, j/16))^2$$

Here we see a correlation between Q_{min} and *sse*, Q_2 and *sse* and between Q_2 (or Q_{min}) and *sse*. In many simulations we saw that Q_2 is a better criterion to assess the estimation and the approximation because in Q_2 more points than the collocation points (with which we minimized Q) are considered. In many simulations we saw that in a bad approximation Q_{min} can be small and Q_2 (or general Q_a) is relative big because the exact solution fulfils the PDE and the boundary value conditions at any point of the approximation area.

If we use a bigger j then we need a bigger k_{max} too because with rising j we compress the bases functions ϕ_{j,k_1,k_2} .

In this example we have (as in other simulations we did) a linear dependency between $ln(Q_{min})$ and $ln(sse)$, between $ln(Q_2)$ and $ln(sse)$ and between $ln(Q_2)$ and $ln(||p - \hat{p}||)$.

This is what we see if we take a look at the regression tables:

***ln(Q_{min})* vs. *ln(sse)*:**

	Estimate	SE	TStat	PValue
1	−2.46106	1.22577	−2.00776	0.0724447
x	1.05934	0.0502733	21.0715	1.28766×10^{-9}

R^2: 0.977974

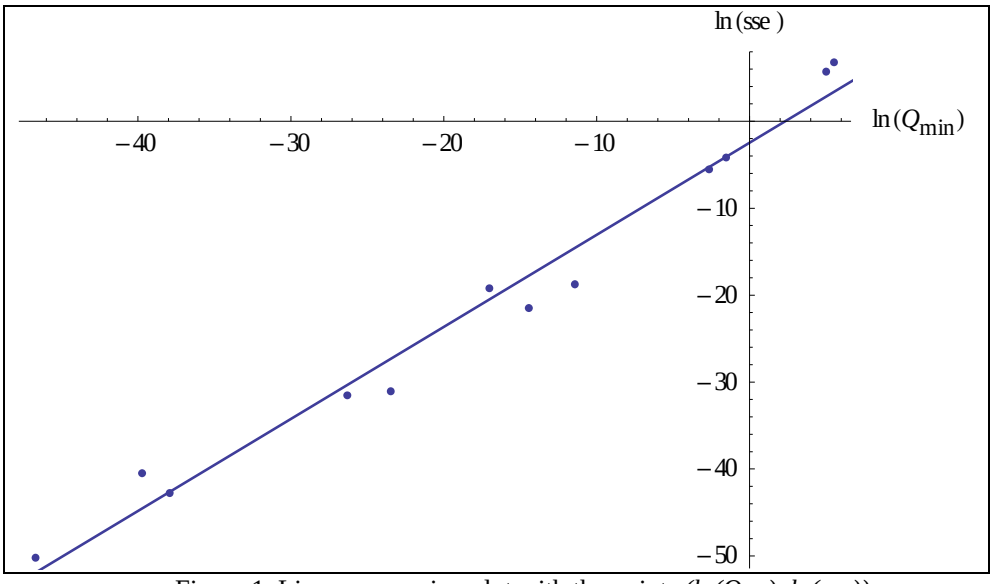

Figure 1. Linear regression plot with the points $(ln(Q_{min}), ln(sse))$

***ln(Q₂)* vs. *ln(sse)*:**

	Estimate	SE	TStat	PValue
1	−3.98319	1.11569	−3.57015	0.00509447
x	1.15459	0.0520054	22.2014	7.71458×10^{-10}

R^2: 0.980115

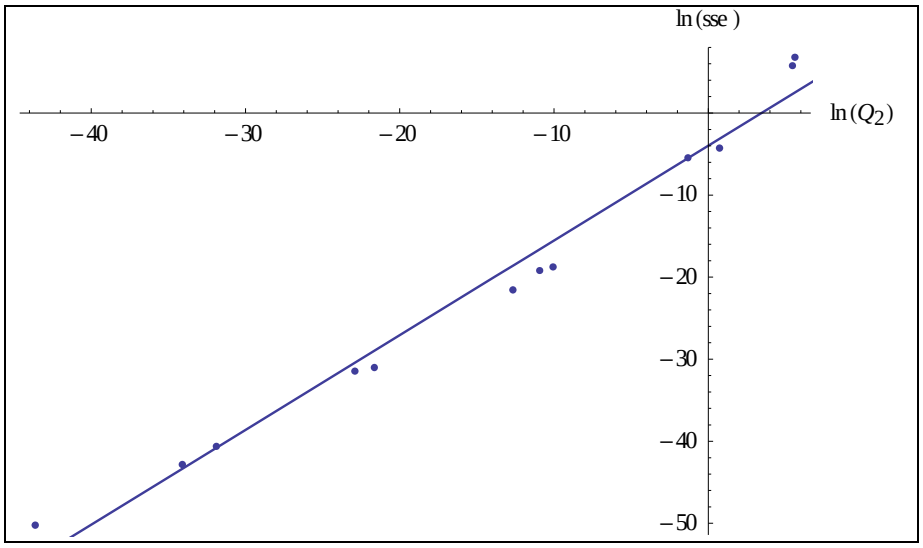

Figure 2. Linear regression plot with the points *(ln(Q_2), ln(sse))*

***ln(Q_2)* and *ln(||p - $\hat{p}$ ||)*:**

	Estimate	SE	TStat	PValue
1	−3.36543	0.453062	−7.42819	0.0000224077
x	0.548946	0.0211184	25.9937	1.63369×10^{-10}

R^2: 0.985416

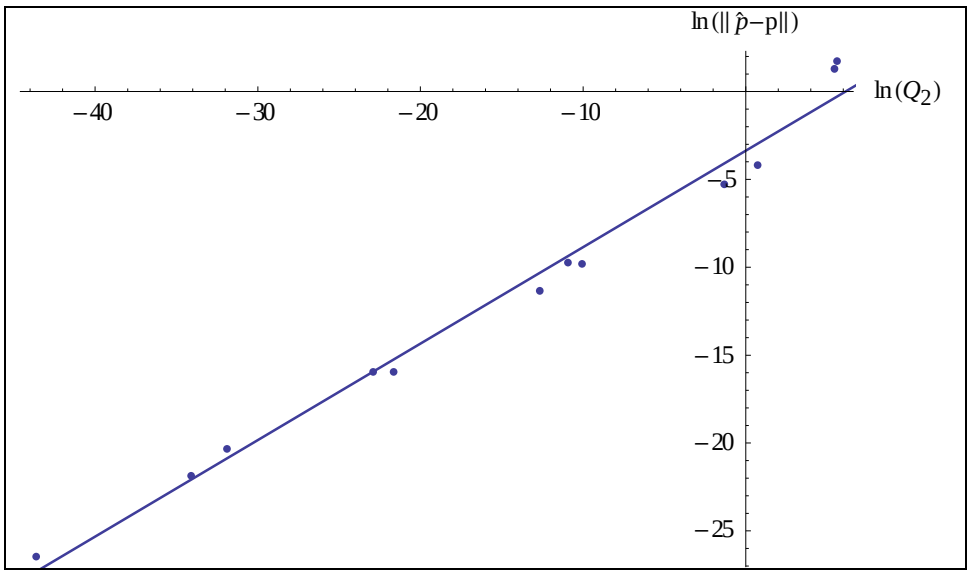

Figure 3. Linear regression plot with the points *(ln(|| $\hat{p}$ – p ||), ln(sse))*

Here is the graph of *u - g* for the best approximation (with the lowest Q_2 and *sse*):

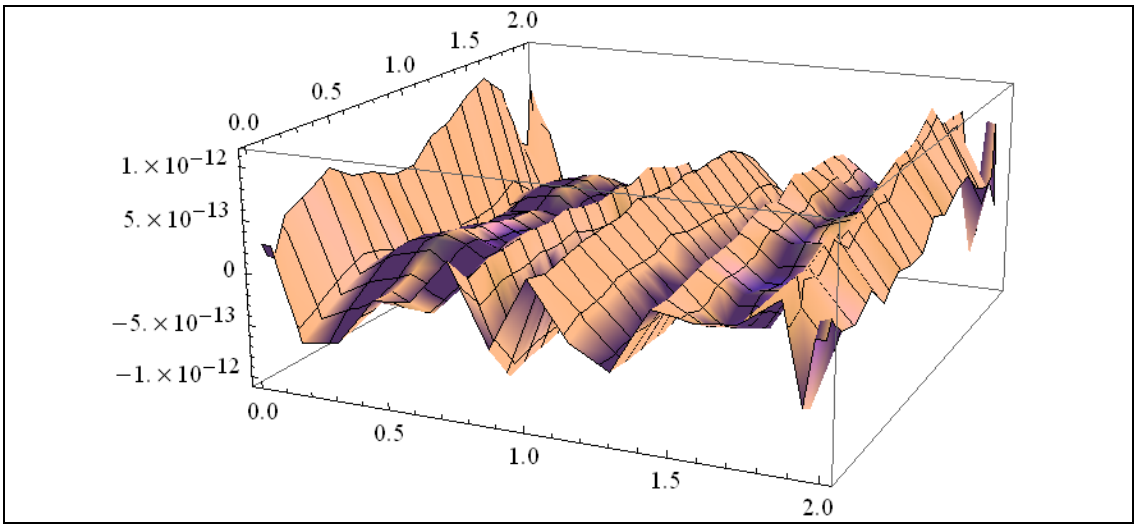

Figure 3**.** Graph of *u - g* for the lowest Q_2 and *sse*

Conclusions

The described wavelet collocation method can be used in different cases and different orders of the PDE (or ODE). The method can even be used if the boundary conditions include parameters and the PDE must not have a special form.

With Q_{min} and Q_2 we can assess an approximation and the parameter estimation. If Q_{min} is too big then the approximation function can not be used. On the other hand we can compare Q_{min} with Q_2 and if Q_2 is too big ($Q_{min} << Q_2$) the approximation also can not be used. In both cases we need to calculate a new minimization. In the first case we do this by using another *j* (or more bases functions if their number is too small). If we have enough basis functions then we can increase *j*. In the second case we need more collocation points.

References

[1] T. S. Carlson, J. Dockery, J. Lund. *A Sinc-Collocation Method for Initial Value Problems.* Mathematics and Computation, Vol. 66, No. 217 (1997)

[2] S. Kameswaran, L. T. Biegler. *Simultaneous Dynamic Optimization Strategies: Recent Advances and Challenges.* Computers and Chemical Engineering, (2006).

[3] A. Nurmuhammada, M. Muhammada, M. Moria, M. Sugiharab. *Double Exponential Transformation in the Sinc-Collocation Method for a Boundary Value Problem with Fourth-Order Ordinary Differential Equation.* Journal of Computational and Applied Mathematics, (2005).

[4] L. Qian. *On the Regularized Whittaker-Koltel'nikov-Shannon Sampling Theorem.* Proceedings of the Amarican Mathematical Society, Vol. 131, No. 4, (2002)

[5] M. Schuchmann. *Approximation and Collocation with Wavelets. Approximations and Numerical Solving of ODEs, PDEs and IEs*. Osnabrück: DAV, (2012).

[6] M. Unser. *Vanishing Moments and the Approximation power of Wavelet Expansions.* Proceedings of the 1996 IEEE International Conference on Image Processing, (1996)

[7] R. Vuduc. *A Wavelet Collocation Method for Solving PDEs.* J. Comp. Phys., (2000).